There Is a 2-Coloring Of $\mathbb{R}^n$ Without a mono Red 2-Stick or a mono Blue Big-Stick

Exposition by William Gasarch-U of MD

Credit Where Credit is Due

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There exists d, for all n, $\mathbb{R}^2 \not\rightarrow (\ell_2, \ell_{2^{dn}})$ (Conlon-Fox, 2017).

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2) restrict COL to S.

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Let $F \colon S \to \binom{[n]}{2q-1}$ by viewing each vector in S as a bit vector though with $\frac{1}{\sqrt{2q}}$ instead of 1.

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Hence $d(u,v) = 2 \times (2q - 1 - f) \times \frac{1}{2q} = \frac{2q-1-f}{q} = 2 - \frac{f+1}{q}$.

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Since $|F(u) \cap F(v)| = q-1$, by the Claim, $d(u,v) = 1$.

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1) One can show there exists d, for all n, $c(n) \geq 2^{dn}$. (KELIN- HAVE WE DONE THAT YET?)

2) Kelin plotted the graph and it seems that $c(n) \geq 2.54 \times 2^{0.266n}$.

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Hence $u + i\vec{1}$ and $v + i\vec{1}$ are both **RED**. They form a **RED** ℓ_2.