There Is a 2-Coloring Of the Plane Without a mono Red 3-Stick or a mono Blue Big-Stick

Exposition by William Gasarch-U of MD

Credit Where Credit is Due

The main result in these slides is due to Conlon and Wu (2022).

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Main Theorem

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Open Find an easier proof of $\mathbb{R}^2$ case.

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And we know
$1 = d(\vec{a}_1, \vec{a}_2)$,
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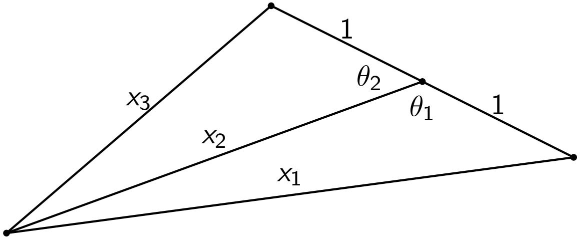
Algebra and Trigonometry



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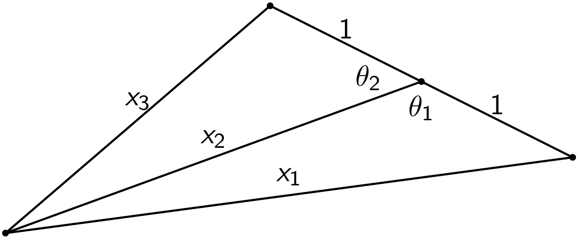
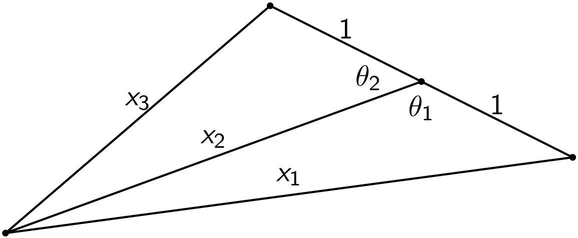


Bottom Triangle:

Algebra and Trigonometry



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Law of cosines: $x_1^2 = x_2^2 + 1 - 2x_2 \cos(\theta_1)$.

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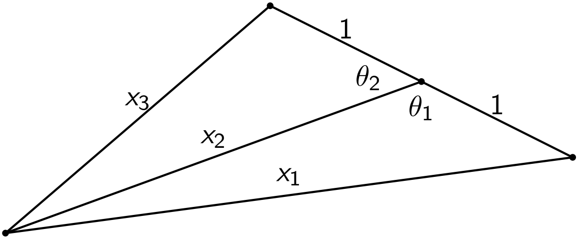

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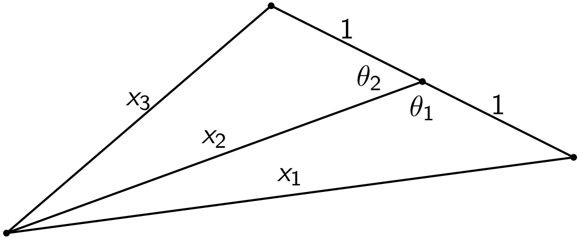

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Add to get

$$x_1^2 + x_3^2 = 2x_2^2 + 2.$$

First Plan On How to Avoid R ℓ_3

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Can get rid of squares.

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We will also have a condition on COL' that will make $\mathrm{COL}(\vec{a}) = \mathrm{COL}'(d(\vec{0}, \vec{a}))$ not have any **B** ℓ_m

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For all $1 \le i \le m$ let $x_i = d(\vec{0}, \vec{a}_i)$.
For all $1 \le i \le m-1$ we know $1 = d(\vec{a}_i, \vec{a}_{i+1})$.

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2) Define $\mathrm{COL} \colon \mathbb{R}^2 \to [2]$ by $\mathrm{COL}(\vec{a}) = \mathrm{COL}'(d(\vec{0}, \vec{a}))$.

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$$\mathrm{COL}'(y) = \mathrm{COL}''(\lfloor y \rfloor \pmod{q}).$$

An Example of A Coloring with $q = 5$

$\mathrm{COL}''(0) = \mathbf{R}$
$\mathrm{COL}''(1) = \mathbf{B}$
$\mathrm{COL}''(2) = \mathbf{B}$
$\mathrm{COL}''(3) = \mathbf{R}$
$\mathrm{COL}''(4) = \mathbf{R}$

An Example of A Coloring with q = 5

$\mathrm{COL}''(0) = $ **R**
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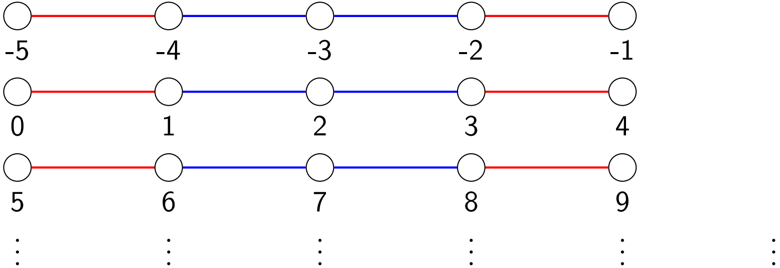

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The next slide recaps where we are and says why COL'' helps us.

COL'' and COL' and COL

Assume $\mathrm{COL}'' \colon \mathbb{Z}_q \to [2]$:

COL'' and COL' and COL

Assume $\mathrm{COL}'' : \mathbb{Z}_q \rightarrow [2]$:
1) COL'' has no **R** solution (in $\mathbb{Z}_q$) to $y_1 + y_3 = 2y_2 + 2$.

COL'' and COL' and COL

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$$\text{For all } 2 \le i \le m-1,\ y_{i-1} + y_{i+1} = 2y_i + 2$$

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Lemmas and a Theorem of Independent Interest

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What does $p(x) = x^2 + \pi x + e \pmod{13}$ mean?

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So it makes sense to consider $p(x) \pmod{q}$ where $p(x) \in \mathbb{R}[x]$.

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X hits at least $q/6$ of the intervals $[0,1)$, $[1,2)$, ..., $[q-1,q)$.

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$\mathrm{SQ}_q = \{1^2 \pmod{q}, 2^2 \pmod{q}, \ldots, q^2 \pmod{q}\}$.

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Case 1: $\mathbf{k} \not\equiv \mathbf{0} \pmod{\mathbf{q}}$

Recap There is a $k \not\equiv 0 \pmod{q}$ such that $|k\alpha \mod q| \le \frac{1}{q}$.

Plan

1) Show $x^2 + \beta \pmod{q}$ hits $\ge (q+1)/2$ intervals.
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Since every element in SQ_q is an integer, hits $(q+1)/2$ intervals.

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Since $k \not\equiv 0 \pmod{q}$, $\{k, 2k, \ldots, qk\} = \{1, 2, \ldots, q\}$. Hence $X = Y$.

Why $\mathbf{m = q^3}$?

We have shown that

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We will do this on the next slide.

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$\{f(k), f(2k), \ldots, f(qk)\}$:
$f(k) = f_1(k) + k\alpha$. **Key** Recall $k\alpha \pmod q)| \leq \frac{1}{q} \leq 1$.
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Recap The set $Y = \{f_1(k), \ldots, f_1(qk)\}$ hits $(q+1)/2$ intervals of length 1.
$Z = \{f(k), \ldots, f(qk)\}$ can be viewed as taking every element in Y and adding or subtracting ≤ 1 to it. It is easy to show that Z hits $\geq q/6$ intervals.

Case 2: $\mathbf{k} \equiv \mathbf{0} \pmod{\mathbf{q}}$

OMITTED FOR NOW.

Another Lemma Of Independent Interest

The Sign Function and Other Notation

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Notation If $\eta \in \{-1, 0, 1\}^*$ then $\eta(i)$ is the ith character in η.

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(x, y)	$(p_1(x, y), p_2(x, y), p_3(x, y))$	sign pattern
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$(10, 0)$	$(7, -27, 40)$	$(+, -, +)$
$(0, 10)$	$(17, 23, -10)$	$(+, +, -)$
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I doubt it has anywhere near 27.

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Lemma About Sign Patterns

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Lemma also a corollary of a theorem by
Olenik-Petrovsky-Thom-Milnor.

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Let $p_1(z_1, z_2)$, . . ., $p_m(z_1.z_2) \in \mathbb{Z}[z_1, z_2]$. We denote this set of polynomials by F.

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3) A **Red image-solution of F** and a **Blue image-solution of F** obvious.

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I am skipping some stuff of interest to get to some stuff that is of more interest.

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prob there is a **blue image-solution** to $p_1, \ldots, p_m$ is
$\le 2500 m^6 b^{m/6}$.

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Number of ways $p_1(a, d), \ldots, p_m(a, d)$ are in the ints is $\le 2500m^6$.

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Hence there exists a 2-coloring with no **R**ℓ_3 or **B**ℓ_m for large enough m.