# BILL, RECORD LECTURE!!!!

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# The Large Ramsey Theorem

**Exposition by William Gasarch**

February 18, 2025

# Notation

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3. $2^A$ is the powerset of $A$.
4. $\binom{A}{a}$ is the set of all $a$-sized subsets of $A$.

Let $\mathrm{COL}\colon \binom{A}{2} \to [2]$. A set $H \subseteq A$ is **homogenous** if $\mathrm{COL}$ restricted to $\binom{H}{2}$ is constant. (From now on **homog**.)

# Large Sets

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$\{101, \ldots, 190\}$ is NOT large.

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$\{1, 2\}$ is always a large homogenous set.
How to change the statement so thats its not stupid?

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$\min(H) = 499$ and $|H| \geq 500$, or
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$\min(H) = 499$ and $|H| \geq 500$, or
$\min(H) = 500$ and $|H| \geq 501$. $H = \{500, \ldots, 1000\}$.
If $\min(H) = 501$ then $H$ cannot be large.

# Proof of the Large Ramsey Thm From The Infinite Ramsey Thm

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The proof will be identical to the proof of

$$\textbf{Infinite Ramsey} \implies \textbf{Finite Ramsey}$$

except at the very end.

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We will use the **Inf Ramsey Theory** to get a contradiction.

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Continue on Next Slide.

# Finite Ramsey VS Large Ramsey

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Proof of Finite Ramsey: The following is a homog set of size $k$:

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Proof of Large Ramsey: The following is a large homog set:

$$H_{\mathrm{LargeSet}} = \{x_1 < \cdots < x_{x_1+1}\}.$$

# Finite Ramsey VS Large Ramsey

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Proof of Finite Ramsey: The following is a homog set of size $k$:

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Proof of Large Ramsey: The following is a large homog set:

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There is an $L$ such that $\mathrm{COL}$ restricted to $H_{\mathrm{LargeSet}}$ is some $\mathrm{COL}_L$.

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This is a contradiction since $\mathrm{COL}_L$ has no large homog sets.

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**STUDENT:** You are telling the same jokes twice, with **Shirley** and **Factorial**.

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Answer on next slide.

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Discuss: Is LR natural?