Monochromatic Unit Squares: Exposition and Open Problems

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1 Introduction

Notation 1.1 Let $a, n \in \mathbb{N}$.

1. $[n] = \{1, \ldots, n\}$.
2. If A is a set then $\binom{A}{a}$ is the set of all a-subsets of A. Hence $\binom{[n]}{2}$ is the complete graph on n vertices.
3. $C_n = (V, E)$ where $V = [n]$ and $E = \{(i, i+1 \pmod{n})\colon 1 \le i \le n\}$. More simply, C_n is the cycle on n vertices.

Definition 1.2 Let $c, d \ge 2$. Let $k \ge 3$.

1. Let $\mathrm{COL}\colon \mathbb{R}^d \to [c]$ be a given coloring. A *monochromatic unit square* (henceforth *mono unit square*) is a 1×1 square in $\mathbb{R}^d$ whose vertices are all the same color.
2. $d(c)$ is the least d such that the following is true: *For all* $\mathrm{COL}\colon \mathbb{R}^d \to [d]$ *there is a mono unit square.*
3. $R_c(C_k)$ is the least n such that, for all $\mathrm{COL}\binom{[n]}{2} \to [c]$ there exists a monochromatic (henceforth mono) cycle of length k.

The following are known:

1. Burr proved that $d(2) \le 6$. He did not publish the result; however, it appears (crediting him) in a paper by Erdös et al. [3]. The proof uses the following theorem: $R_2(C_4) = 6$. The accounts of Burr's result that we have seen say that $R_2(C_4) = 6$ is either *well-known* or *easy* and do not give a reference, which is Chvátal & Harary [2]. It may be well-known and easy for some people; however, when I teach high school students Burr's result they do not consider $R_2(C_4) = 6$ to be well known or easy.
2. $d(2) \le 5$. All accounts of this say it follows easily from Burr's proof that $d(2) \le 6$. This may be true for some people; however, when I teach high school students it is not clear how to present the result to them.

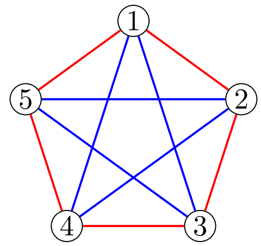


Figure 1: $R_2(C_4) \geq 6$

3. $d(2) \leq 4$. This proof is very different from the proofs of $d(2) \leq 6$ and $d(c) \leq 5$.

In this article we do the following:

1. Present the complete proof of $d(2) \leq 6$ and $d(c) \leq 5$ including the parts that are allegedly easy.

2. Present bounds on $d(c)$. These results appear to be new.

3. Present some open problems. Some of them are motivated by presenting these theorems, and others, to high school students.

Convention 1.3 We use **R** and **B** for the actual colors. So we might say COL$(1, 2) =$ **R**. We use the terms *red* and *blue* in pros. So we might say *Since* COL$(1, 2) =$ **B** *we have a blue* C_4. The symbol **R** (**B**) will appear red (blue) if you are reading this paper in color and in normal font (black) if you are not.

2 $d(2) \leq 6$

2.1 Lemma on Mono C_4

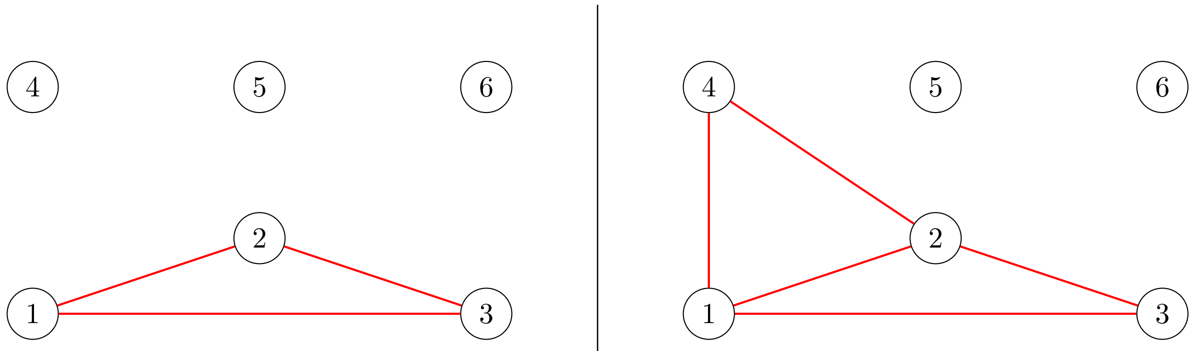
Theorem 2.1 $R_2(C_4) = 6$.

Proof:
1) $R_2(C_4) \geq 6$:

Figure 1 is a 2-coloring of $\binom{[5]}{2}$ with no mono C_4. (If you are reading this in black and white instead of color then the coloring is that the cycle $1-2-3-4-5$ is all red edges and the rest of the edges are blue.)

We present a COL: $\binom{[5]}{2} \rightarrow [2]$ with no mono C_4.

Note 2.2 The coloring in Figure 1 has a mono C_5 but not a mono C_4. The study of $R_c(C_k)$ is very different from the usual Ramsey's theorem where one seeks mono K_k's in that if you have a mono K_k then you have a mono K_{k-1} (and lower) but if you have a mono C_k you need not have a mono C_{k-1}.



(a) Red Triangle

(b) $\deg_{\mathbf{R}}(v) \ge 2$ yields a mono C_4

Figure 2

2) $R_2(C_4) \le 6$:

Let COL: $\binom{[6]}{2} \rightarrow [2]$. By standard Ramsey Theory there is a mono K_3. We assume it is red and on the vertices $\{1,2,3\}$. See Figure 2a.

We view $\{1,2,3\}$ and $\{4,5,6\}$ as the sides of a bipartite graph.

Notation 2.3

1. If $a \in \{1,2,3\}$ then $\deg_{\mathbf{R}}(a)$ is the number of red edges between a and $\{4,5,6\}$.
2. If $a \in \{4,5,6\}$ then $\deg_{\mathbf{R}}(a)$ is the number of red edges between a and $\{1,2,3\}$.

There are cases. Each case assumes the negation of the prior ones.

Case 1 $\exists v \in \{4,5,6\}$, $\deg_{\mathbf{B}}(v) \le 1$. The situation is pictured in Figure 2b. Note that $1-4-2-3-1$ is a mono C_4.

Case 2 $(\exists v \in \{4,5,6\})[\deg_{\mathbf{B}}(v) = 3]$. The situation is pictured in Figure 3a.

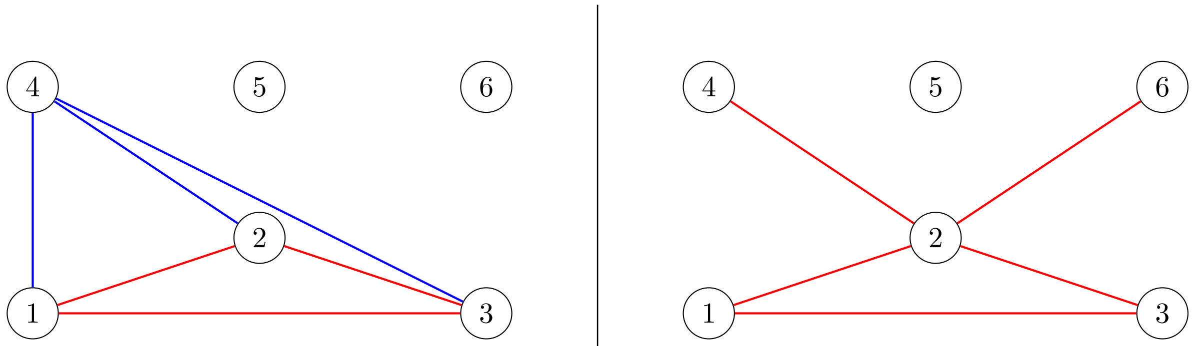
From the negation of Case 1, $\deg_{\mathbf{B}}(5) \ge 2$.

1. If $\mathrm{COL}(5,1) = \mathrm{COL}(5,2) = \mathbf{B}$ then there is a blue C_4: $5-1-4-2-5$.
2. If $\mathrm{COL}(5,2) = \mathrm{COL}(5,3) = \mathbf{B}$ then there is a blue C_4: $5-3-4-2-5$.
3. If $\mathrm{COL}(5,1) = \mathrm{COL}(5,3) = \mathbf{B}$ then there is a blue C_4: $5-1-4-3-5$.

Case 3 $(\exists v \in \{1,2,3\})[\deg_{\mathbf{R}}(v) \ge 2]]$

The situation is pictured in Figure 3b.

1. If $\mathrm{COL}(1,4) = \mathbf{R}$ then there is a red C_4: $1-4-2-3-1$.
2. If $\mathrm{COL}(3,6) = \mathbf{R}$ then there is a red C_4: $3-6-2-1-3$.
3. If $\mathrm{COL}(3,4) = \mathbf{R}$ then there is a red C_4: $3-4-2-1-3$.



(a) $\deg_{\mathbf{B}}(4) = 3$

(b) $\deg_{\mathbf{R}}(v) \geq 2$ yields a mono C_4

Figure 3

4. If $\mathrm{COL}(1,6) = \mathbf{R}$ then there is a red C_4: $1 - 6 - 2 - 3 - 1$.
5. If all of those edges are blue then there is a blue C_4: $1 - 4 - 3 - 6 - 1$.

Case 5 The negation of Cases 1,2,3,4. So we have the following:

1. $(\forall v \in \{1,2,3\})[\deg_{\mathbf{R}}(v) = 1]$.
2. $(\forall v \in \{4,5,6\})[\deg_{\mathbf{R}}(v) = 1]$.
3. Hence we can assume
   (a) $\mathrm{COL}(1,4) = \mathrm{COL}(2,5) = \mathrm{COL}(4,6) = \mathbf{R}$
   (b) All other edges between $\{1,2,3\}$ and $\{4,5,6\}$ are blue. (We will find some other edges that must be blue.)

The situation is pictured in Figure 4. If any of $(4,5)$, $(5,6)$,or $(4,6)$ are $\mathbf{R}$ then there will be a red C_4. Hence they are all blue. Hence $4 - 5 - 6 - 2 - 4$ is a blue C_4.

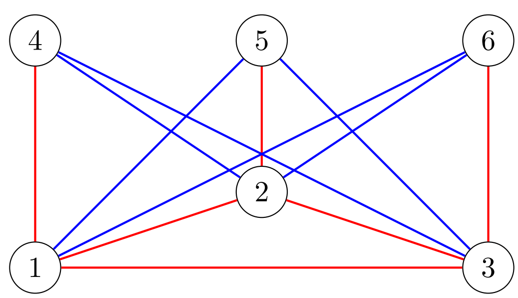


Figure 4: Negation of Cases 1,2,3,4

▮

Open Problem 2.4 The proof of Theorem 2.1 took five cases. Some of the cases had subcases. Is there a proof with less cases. Perhaps beginning with the fact that any 2-coloring of the edges of K_6 has *two* mono triangles.

2.2 Proof of the $\mathbb{R}^6$ Theorem

Theorem 2.5 $d(2) \le 6$.

Proof:

Let COL: $\mathbb{R}^6 \rightarrow [2]$. We form a coloring COL$'$: $\binom{[6]}{2} \rightarrow [2]$.

Let

$p_{1,2} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0, 0)$.

$p_{1,3} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 0, 0)$.

$\vdots \quad \vdots \quad \vdots$

$p_{5,6} = (0, 0, 0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Define COL$'(i, j) =$ COL$(p_{i,j})$.

By Theorem 2.1 there exists a mono C_4. Let the vertices be a, b, c, d and the color be red. Then

$$\mathrm{COL}'(a, b) = \mathrm{COL}'(b, c) = \mathrm{COL}'(c, d) = \mathrm{COL}'(d, a) = \mathbf{R}$$

hence

$$\mathrm{COL}(p_{a,b}) = \mathrm{COL}(p_{b,c}) = \mathrm{COL}(p_{c,d}) = \mathrm{COL}(p_{d,a}) = \mathbf{R}.$$

It it easy to see that

$$d(p_{a,b}, p_{b,c}) = d(p_{b,c}, p_{c,d}) = d(p_{c,d}, p_{d,a}) = 1.$$

Hence we have a mono unit square. ∎

AUGUSTE- IS IT THE CASE THAT IF YOU HAVE 4 POINTS p, q, r, s IN $\mathbb{R}^n$ AND $d(p, q) = d(q, r) = d(r, s) = d(s, p) = 1$ THEN THE FOUR FORM A UNIT SQUARE?

The proof of Theorem 2.5 did not use any geometry. That makes it easy to teach since the students do not have to visualize $\mathbb{R}^6$.

3 $d(2) \le 5$

Definition 3.1

1. For $1 \le i < j \le 6$ let $p_{i,j}$ be as in Theorem 2.5.
2. $P = \{p_{i,j} : 1 \le i < j \le 6\}$.
3. $H = \{(x_1, \ldots, x_6) \in \mathbb{R}^6 : x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = \frac{2}{\sqrt{2}}\}$.

We state the following easy fact for reference.

Fact 3.2

1. *For every* COL: $P \rightarrow [2]$ *there exists a mono unit square. This follows from the proof of Theorem 2.5.*

2. $P \subseteq H$.

3. H is a 5 dimensional hyperplane. Hence there is a rotation that maps H to $\mathbb{R}^6$. By omitting the last coordinate (which is always 0) there is a rotation f that maps H to $\mathbb{R}^5$.

4. f is a bijection from H to $\mathbb{R}^5$. Let g be its inverse.

5. f preserves distance: $d(x, y) = d(f(x), f(y))$.

AUGUSTE- FIND THE ACTUAL ROTATION. I ASSUME ITS A MATRIX.

Theorem 3.3 *For all* $\mathrm{COL}\colon \mathbb{R}^5 \to [2]$ *there exists a mono unit square.*

Proof: We define $\mathrm{COL}'\colon P \to [2]$ by

$$\mathrm{COL}'(p_{i,j}) = \mathrm{COL}(g(p_{i,j})).$$

By Fact 3.2.1 there is a mono unit square (using COL') in $\mathbb{R}^6$ using the vertices $p_{a,b}, p_{b,c}, p_{c,d}, p_{d,a}$. By Fact 3.2.4 and 5 the points $g(p_{a,b}), g(p_{b,c}), g(p_{c,d}), g(p_{d,a})$ form a mono unit square (using COL) in $\mathbb{R}^5$.

∎

The proof of Theorem 3.3 did not use any geometry. That makes it easy to teach since the students do not have to visualize $\mathbb{R}^5$.

4 $d(2) \leq 4$

The following theorem was proven by Kent Cantwell [1].

Theorem 4.1 *For all* $\mathrm{COL}\colon \mathbb{R}^4 \to [2]$ *there exists a mono unit square.*

AUGUSTE- LOOK AT THE PROOF AND SEE IF WE CAN SKETCH IT OR PRESENT IT OR WHAT. ALSO SEE IF THE NEXT STATEMENT IS TRUE

The proof of Theorem 4.1 uses geometry. This makes it hard to teach.

Open Problem 4.2 Give a proof that $d(2) \leq 4$ that uses less geometry.

5 What If We Use More Colors?

Theorem 5.1

1. $d(c) \leq R_c(4)$.
2. $d(c) \leq R_c(4) - 1$.

Proof:
1) $d(c) \le R_c(4)$.

Let COL: $\mathbb{R}^{R_c(4)} \to [c]$. We form a coloring COL$'$: $\binom{[R_c(4)]}{2} \to [c]$.

We define $\binom{R_c(4)}{2}$ points in $\mathbb{R}^{R_c(4)}$.

$p_{1,2} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0)$.

$p_{1,3} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, \ldots, 0)$.

$\vdots \quad \vdots \quad \vdots$

$p_{R_c(4)-1,R_c(4)} = (0, \ldots, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Define COL$'(i, j) = $ COL$(p_{i,j})$.

By the definition of $R_c(4)$ there exists a mono C_4. The rest of the proof is identical to the proof of Theorem 2.5.

2) In Theorem 3.3 we obtained $d(2) \le 5$ from the proof of $d(2) \le 6$. The same technique can be used to obtain a proof of $d(c) \le R_c(C_4) - 1$ from the proof that $d(c) \le R_c(C_4)$. ∎

The proof of Theorem 5.1 did not use any geometry. That makes it easy to teach since the students do not have to visualize high and unknown dimensions.

To use Theorem 5.1 we need to know upper bounds on $R_c(C_4)$. The following lemma is obtained from a variety of results in Section 6.3.2 of Radziszowski's survey of small Ramsey numbers [4]. Go there for references.

Lemma 5.2

1. $R_3(C_4) = 11$.
2. $R_4(C_4) = 18$.
3. *For all* $c \ge 1$, $R_c(C_4) \le c^2 + c + 1$.
4. *For all* $c \ge 2$, c *even,* $R_c(C_4) \le c^2 + c$.

By combining Theorem 5.1 and Lemma 5.2 we obtain the following.

Theorem 5.3

1. $d(3) \le 10$.
2. $d(4) \le 17$.
3. *For* $c \ge 5$ $d(c) \le c^2 + c$.
4. *For* $c \ge 6$, c *even,* $d(c) \le c^2 + c - 1$.

Open Problem 5.4

1. Find better upper bounds on $d(c)$. This may require proofs similar to that of Theorem 4.1 and hence geometry.

   AUGUSTE- TRY TO DO THIS

2. Find lower bounds on $d(c)$ by finding colorings with no mono unit.

6 What If We Want a Monochromatic Unit k-Gon?

Definition 6.1 Let $c, d \geq 2$. Let $k \geq 3$.

1. Let $\mathrm{COL}\colon \mathbb{R}^d \to [c]$ be a given coloring. A *mono unit k-gon* is a regular k-gon with all sides of length 1. whose vertices are all the same color.

2. $d_k(c)$ is the least d such that the following is true: *For all* $\mathrm{COL}\colon \mathbb{R}^d \to [d]$ *there is a mono unit k-gon.*

3. Recall that $R_c(C_k)$ is the least n such that, for all $\mathrm{COL}\binom{[n]}{2} \to [c]$ there exists a mono cycle of length k.

The proof of the following Theorem is similar to that of Theorem 5.1 and hence is omitted.

Theorem 6.2

1. $d_k(c) \leq R_c(k)$.

2. $d_k(c) \leq R_c(k) - 1$.

To use Theorem 6.2 we need to know upper bounds on $R_c(C_k)$. See Radziszowski's survey of small Ramsey numbers [4] for such bounds.

7 Acknolwedgements

We would like to thank (1) Dömötör Pálvölgvi for pointing us to Kent Cantwell's paper that proved $d(2) \leq 4$, and (2) Alexander Soifer who encourages us to make an article out of our thoughts.

References

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