Schur's Thm + FLT implies Primes Infinite

May 10, 2025

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1. All of these proofs are harder than the usual proof
2. All of these proofs have other points to make after they prove primes ∞.

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2. Look at what it means to ask the question in domains other than $\mathbb{N}$. (In fact, asking it over $\mathbb{N}$ is not quite right).
3. Look at domains where the number of primes is finite and see where the standard proof fails, and where the EG-proof fails.

Background Needed For EG-Proof

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2. $\binom{A}{k}$ is the set of all subsets of A of size k.

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There exists a COL'-homog set H of size 3 (thats all we need!).
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So let $S(c) = R(3; c)$ (homog set 3, colors c).

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In modern terminology:

$$(\forall n \geq 3)(\forall x, y, z \in \mathbb{N} - \{0\})[x^n + y^n \neq z^n].$$

This has come to be known as **Fermat's Last Theorem**.

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2) Andrew Wiles and Richard Taylor proved FLT in the early 1990s with techniques far beyond what Fermat could have known.

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To the margin add 200 pages.

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$(p_1^{x_1} \cdots p_L^{x_L})^4 + (p_1^{y_1} \cdots p_L^{y_L})^4 = (p_1^{z_1} \cdots p_L^{z_L})^4$
This violates FLT for $n = 4$.

How to Ask the Question of Primes Infinite

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2) $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$.

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On theses slides **infinite** will mean **infinite up to units**.

The Normal Proof that Primes are Infinite and Where it Falls Apart

May 10, 2025

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   $p \notin \{p_1, \ldots, p_n\}$ since if it was then $0 \equiv 1 \pmod{p}$.

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Upshot The proof that $\mathbb{Z}$ has an infinite number of primes uses that, for all $p_1 \cdots p_n + 1$ is never a unit.

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$\mathbb{Q}_2$ has a Finite Number of Primes

We actually have a list of primes: $\{2\}$.
$N = 2 + 1 = 3$ which is a unit.
So similar to why the proof fails for $\mathbb{Q}$.

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So there are no primes.

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   This is where the proof breaks down! In $\mathbb{AI}$ you can keep going down and never get to a prime.

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Def An **Atomic Integral Domain** is an integral domain such that every element of $\mathbb{D} - (\mathbb{U} \cup \{0\})$ can be written (not necessarily uniquely) as $up_1^{x_1} \cdots p_m^{x_m}$ where u is a unit and all of the p_i's are irreducible.

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Upshot The proof that $\mathbb{Z}$ has an infinite number of primes used that $\mathbb{Z}$ is atomic.

The EG-Proof that Primes are Infinite and Where it Falls Apart

May 10, 2025

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Not True Fix n. Let $u_x = u_y = \frac{1}{2}$, $u_z = 1$, $X = Y = Z = 1$.

$$u_x X^n + u_y Y^n = u_z Z^n$$

Becomes

$$\frac{1}{2}1^n + \frac{1}{2}1^n = 1 \times 1^n$$

$$\frac{1}{2} + \frac{1}{2} = 1$$

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Its because the following **variant** of FLT is false for $\mathbb{Q}$:

There exists $n \in \mathbb{N}$ such that the following has no solution:

$$u_x X^n + u_y Y^n = u_z Z^n$$

where $u_x, u_y, y_z \in \mathbb{U}$ and $X, Y, Z \in \mathbb{Q}$.

Project TO-DO List

May 10, 2025

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2. Read in Gasarch's paper the **Sanity Check** which has more domains with a finite number of primes.
3. Read the other papers on the website of Ramsey-Primes paper. Some of the papers are difficult so try to just figure out the proof for $\mathbb{Z}$ or $\mathbb{N}$, and then see where it fails for $\mathbb{Q}$ and $\mathbb{Q}_2$. (I think they all fail for $\mathbb{AI}$ because $\mathbb{AI}$ is not atomic, though check that.)