BILL, RECORD LECTURE!!!!

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Euclidean Ramsey Theory: Triangles

Exposition by William Gasarch

May 3, 2025

Credit Where Credit is Due

The the main thm of these slides is due to
Paul Erdös, Ronald Graham, Peter Montgomery, Bruce L. Rothchild, Joel Spencer, Ernst G. Straus.

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Euclidean Ramsey Theorems I

Journal of Combinatorial Theory (A), Vol. 14, 341-363, 1973

Here is a link.
`https://www.cs.umd.edu/~gasarch/TOPICS/eramsey/eramseyOne.pdf`

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Answer on next slide

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New Question either a mono $1-1-1$ **or** mono $2-2-2$ **or** $\cdots$.

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We prove this rather than $T_1 - T_{\sqrt{3}} - T_2$ since this makes the figures easier to draw.

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Assume by way of contradiction that there is a $\mathrm{COL}\colon \mathbb{R}^2 \to [2]$ with no mono T_2, $T_{2\sqrt{3}}$ or T_4.

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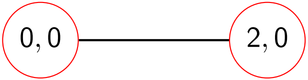

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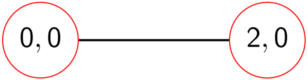

On the next slide we add four more points of interest.

Six Point of Interest

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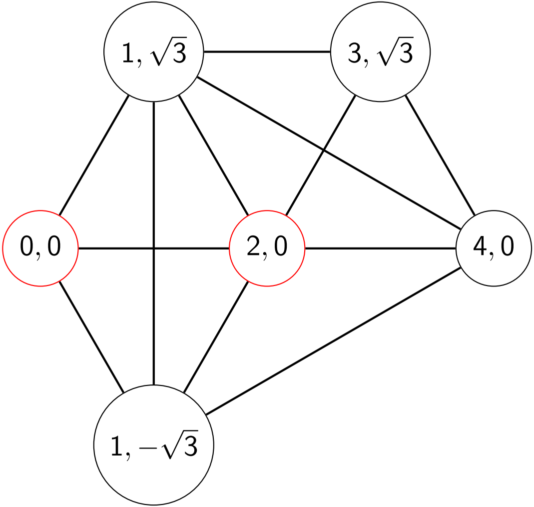

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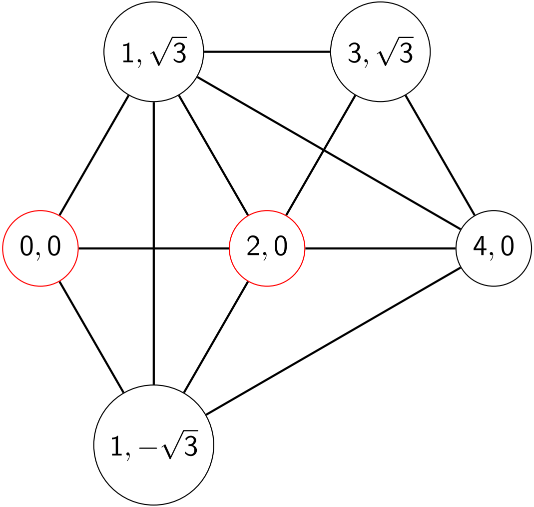

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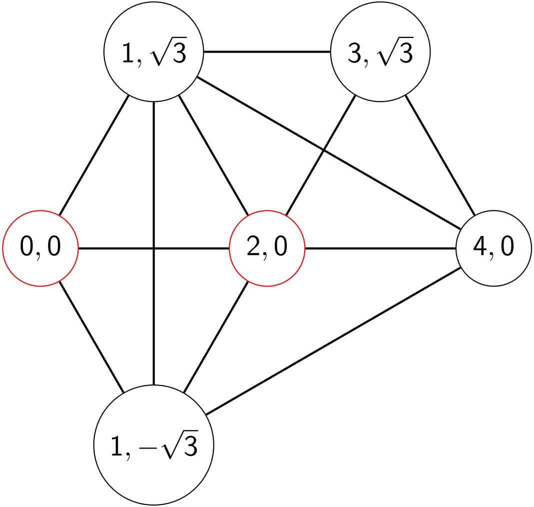

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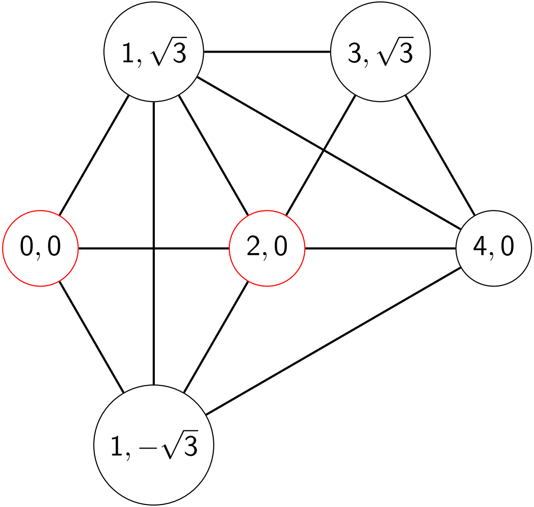

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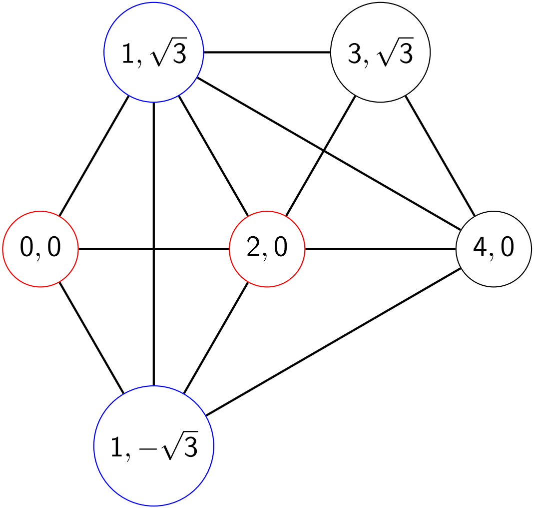
Next picture has this information.

$(1, \sqrt{3})$ and $(1, -\sqrt{3})$ are B



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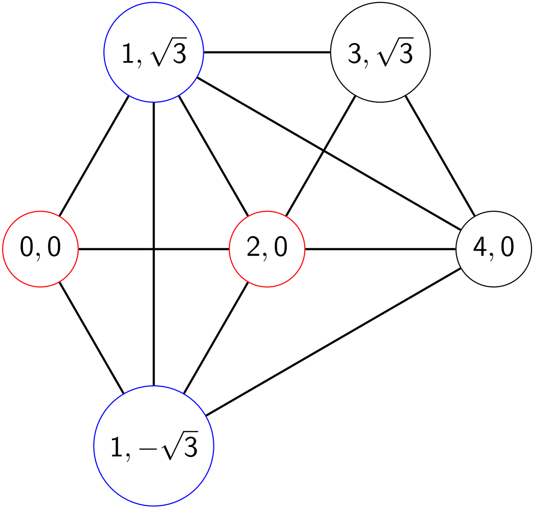

$(1, \sqrt{3}) - (1, -\sqrt{3}) - (4, 0)$ is a $T_{2\sqrt{3}}$ so $\mathrm{COL}(4, 0) =$ **R**.

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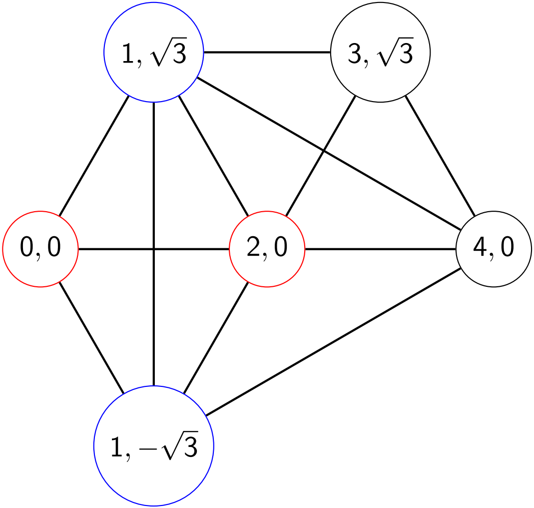

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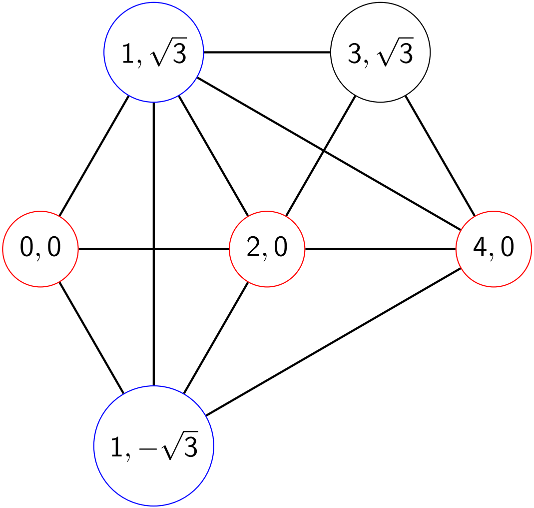

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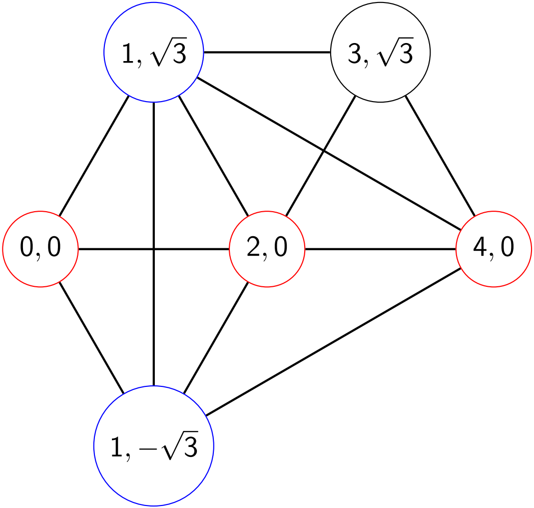

$(2, 0) - (4, 0) - (3, \sqrt{3})$ is a T_2 so $\mathrm{COL}(3, \sqrt{3}) =$ **B**.

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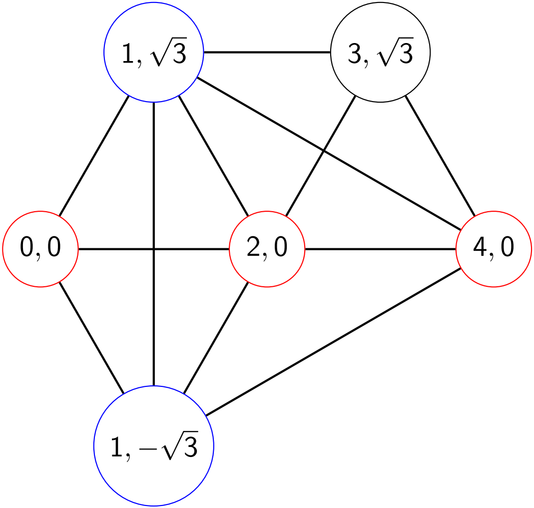

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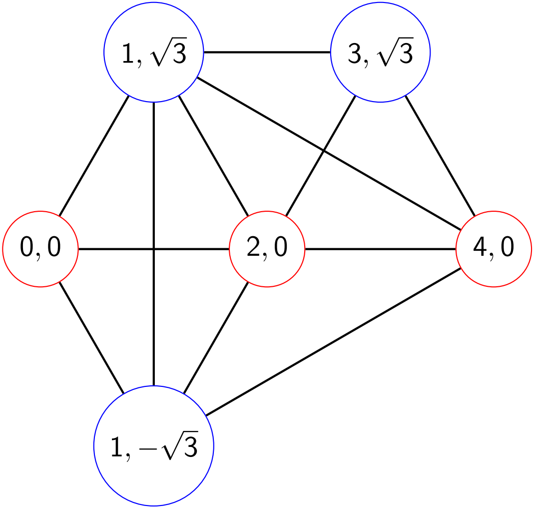

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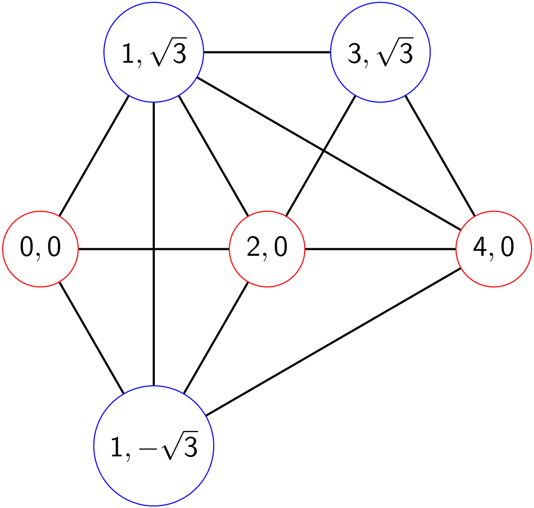
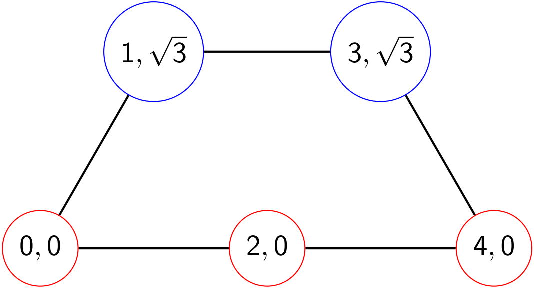


Next picture removes stuff we don't need anymore.

Where We Are Now



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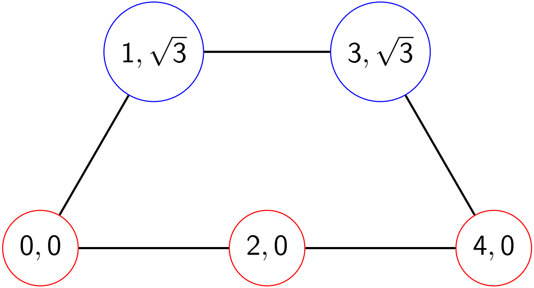
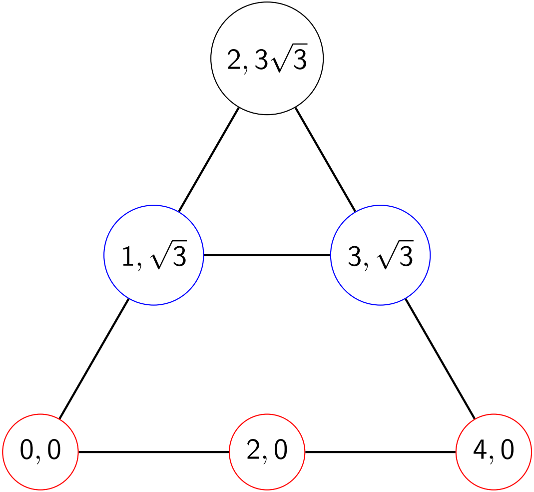


We add the point $(2, 2\sqrt{3})$ on the next slide.

A Point That Can't be R or B



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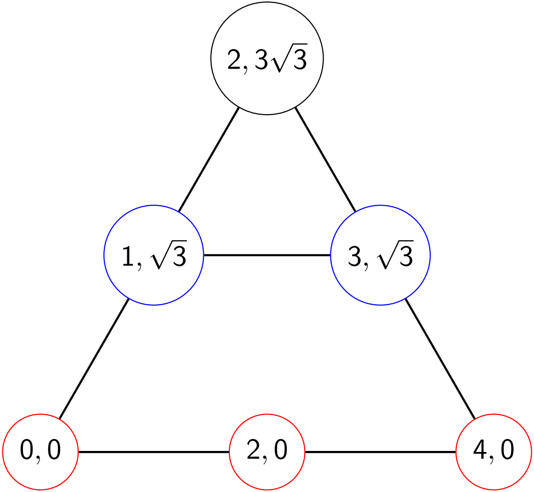

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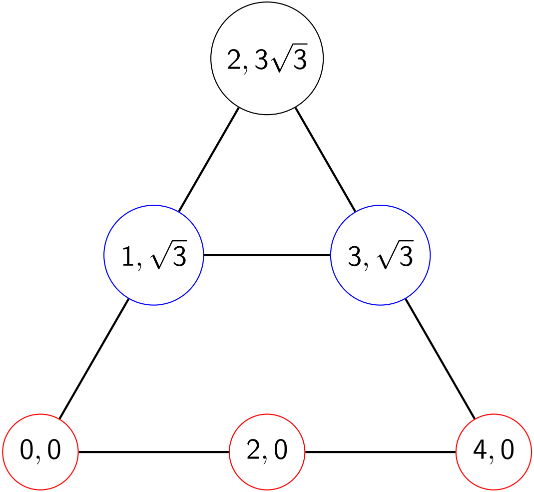

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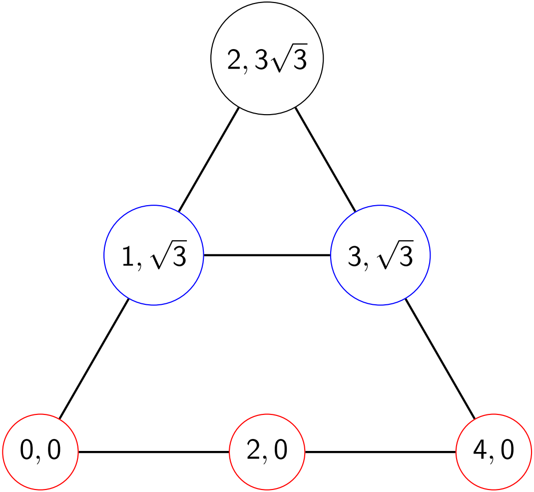

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$\mathrm{COL}(2, 3\sqrt{3}) \notin \{$**R**, **B**$\}$. Contradiction!

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2) Could generalize to any $c \in \mathbb{N}$ and any finite set of $\mathbb{R}^+$
3) Since its a lot of case work, maybe programming would help.