BILL, RECORD LECTURE!!!!

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A Variant on $R(3) = 6$

Exposition by William Gasarch

April 3, 2025

Credit Where Credit Is Due

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The questions raised in these slides are due to Paul Erdös.

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Joel Spencer.

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Vote: YES or NO or UNKNOWN TO SCIENCE.

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Answer on next slide.

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We will sketch Spencer's proof.

G **Such That**

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G **Has No** K_4 **Subgraph,**

G **Has** 3×10^8 **Vertices**

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A Condition That Implies $\mathrm{RAM}(G)$

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4. x, y, z will be vertices.
5. $xy = \{x, y\}$, $xyz = \{x, y, z\}$.

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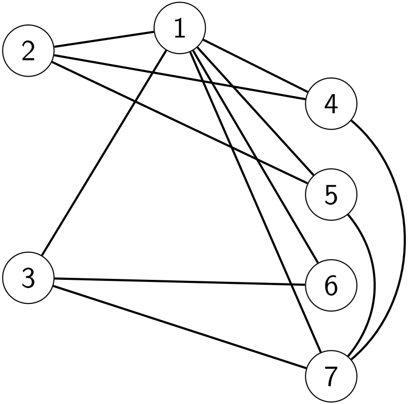

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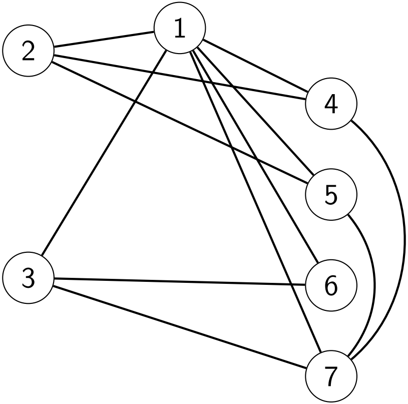

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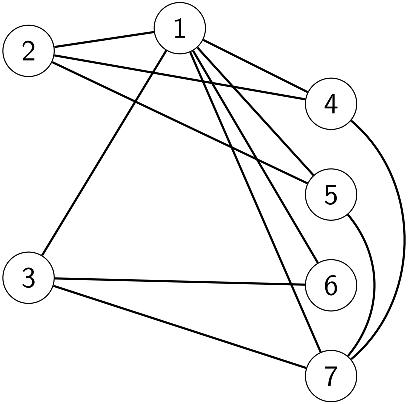

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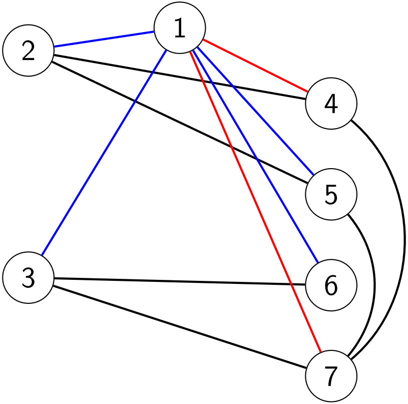

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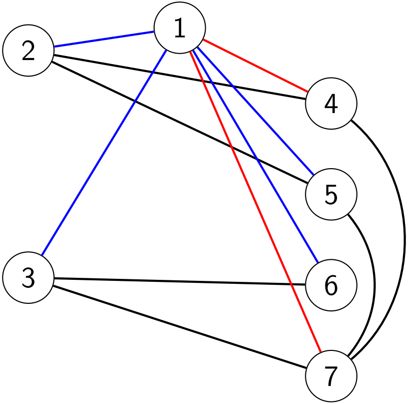

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Relating $|U|$ and $|U^{\mathrm{COL}}|$

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$|U|$ which does not depend on COL, only on G.

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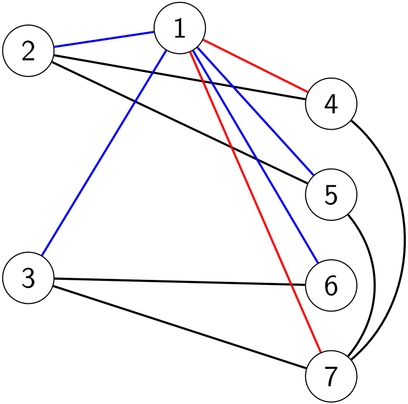

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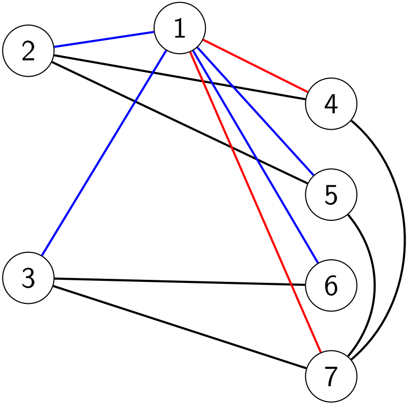

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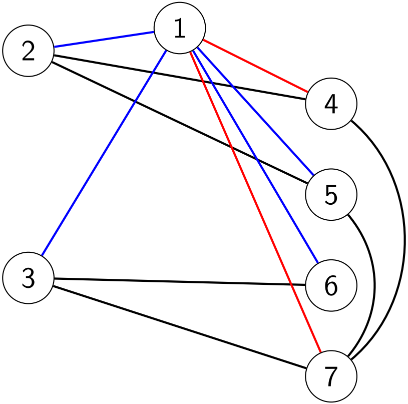

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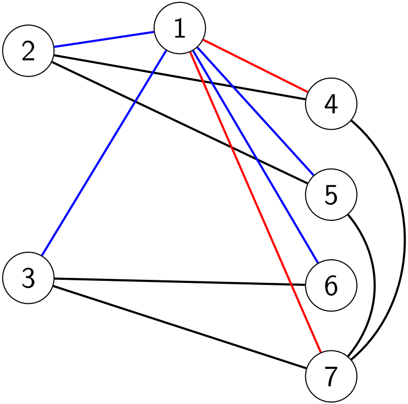

$U^{\mathrm{COL}}(1) = \{(1, 124), (1, 137), (1, 157)\}$.
Let $\mathbf{R} = \{4, 7\}$, $\mathbf{B} = \{2, 3, 5, 6\}$.
$|U^{\mathrm{COL}}(1)| = |\{(x, y) \colon x \in \mathbf{R} \wedge y \in \mathbf{B} \wedge 1xy \in T\}|$.

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The above statements are obvious.

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Note $|U^{\rm COL}(x)| \le A(x)$ relates

$|U^{\rm COL}(x)|$ which depends on $\rm COL$, and

Motivated Def About Max Over Partitions

Recall from last slide:

$|U^{\mathrm{COL}}(x)| \le \max_{N(x)=R\cup B} |\{(y,z) \in E \colon y \in R \wedge z \in B\}|$.

This inequality motivates the following definition.

Apologies to the Math Majors who are not used to motivations.

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$A(x)$ which does not depend on COL, only on G.

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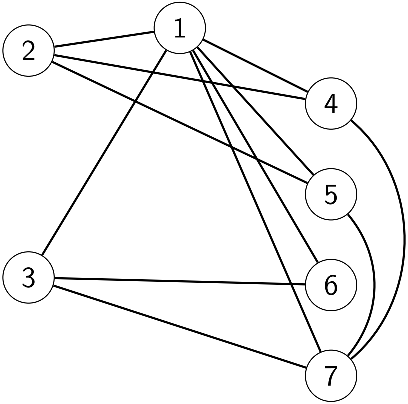

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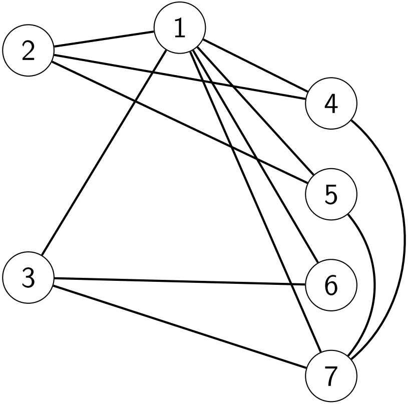

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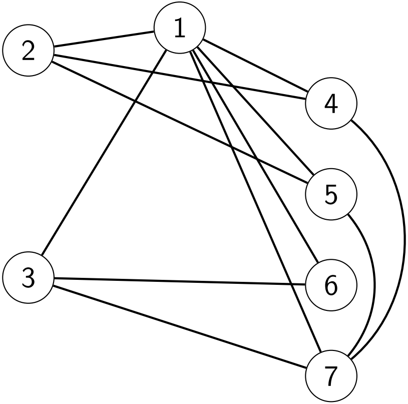

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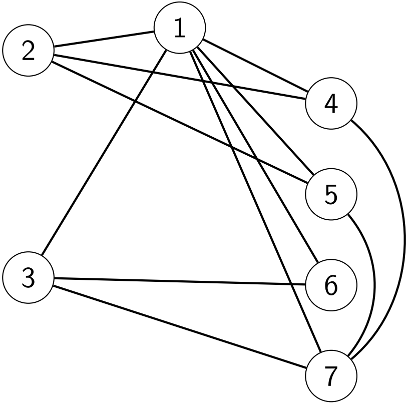

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The next slide is just the neighbors of 1.

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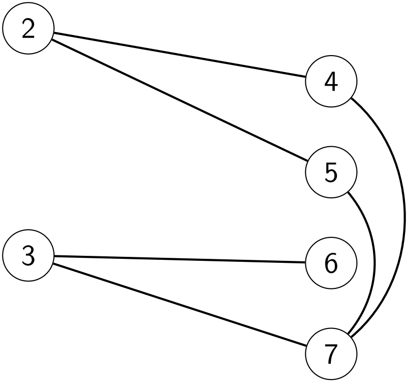

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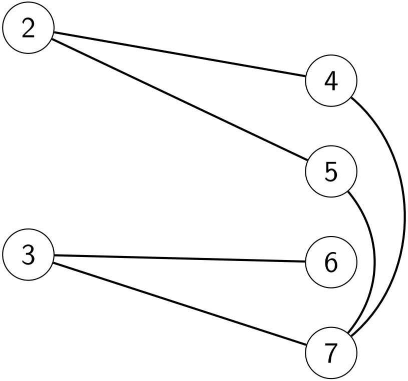

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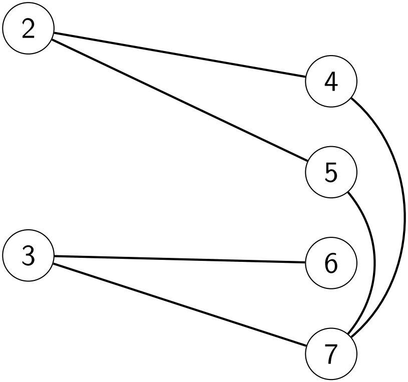

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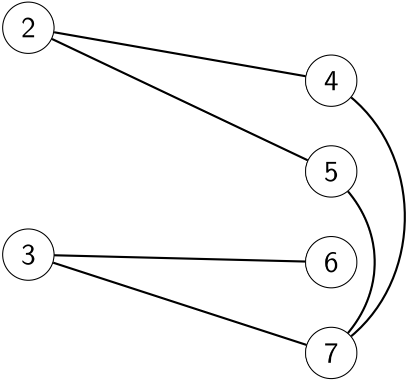

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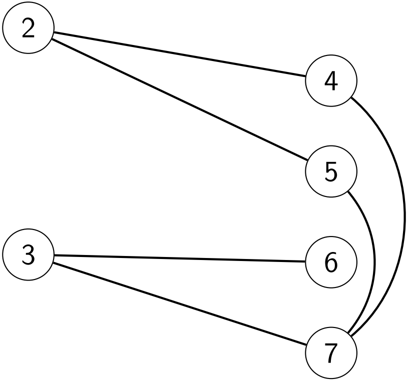

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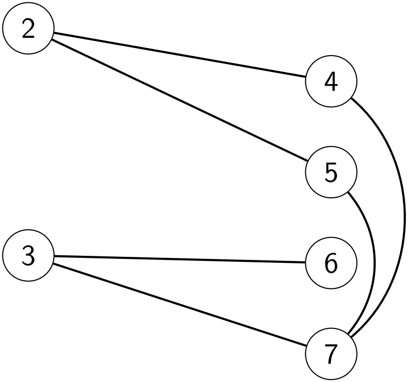

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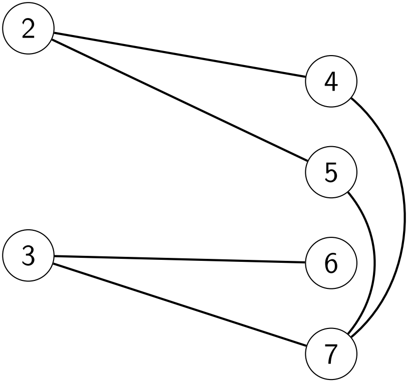

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One can easily show $A(1) \leq 4$, so $A(1) = 4$.

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So we have a way to test if $\mathrm{RAM}(G)$ that is much faster than looking at every 2-coloring of E.

$\exists G$ **Such That**

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We show that for appropriate p, n the prob that $G(n, p)$ has the properties we want is nonzero.

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2) We pick p, n so $\mathrm{Prob}(\sum_{x \in V} A(x)) < \frac{2}{3}|U|) > 0$.

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That is the desired G.

Project Idea
Taking Probability Seriously

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Looking at the proof we are inspired to try probability.

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5) Call the new graph G. Compute $\sum_{x \in V} A(x)$ and $\frac{2}{3}|U|$. If $\sum_{x \in V} A(x) < \frac{2}{3}|U|$ then YEAH!

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There may be many optimizations to do on the pseudo-code I wrote. The list of K_4s- is there a way to delete those you don't need to deal with?