BILL, RECORD LECTURE!!!!

BILL RECORD LECTURE!!!

Euclidean Ramsey Theory: Area

Exposition by William Gasarch

May 1, 2025

Mono Triangles

Def Assume there is a coloring of $\mathbb{R}^2$. A **Mono Triangle** is a triangle with all three vertices the same color.

Mono Triangles

Def Assume there is a coloring of $\mathbb{R}^2$. A **Mono Triangle** is a triangle with all three vertices the same color.

We will prove the following:
Thm $\forall$ finite colorings of $\mathbb{R}^2$, $\exists$ a mono triangle with area 1.

The Two Color Case

The 2-Color Case

Thm For all $\mathrm{COL}\colon \mathbb{R}^2 \to [2]$ there is a mono triangle with area 1.

The 2-Color Case

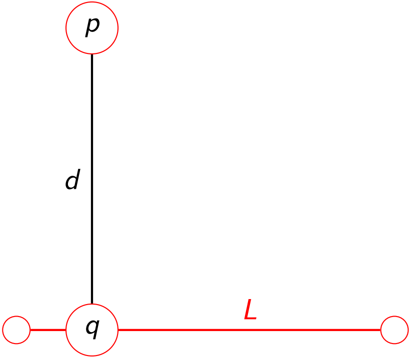
Thm For all $\mathrm{COL}\colon \mathbb{R}^2 \to [2]$ there is a mono triangle with area 1.
Case 1: $\exists$ a horiz. line L which is all **R**, and a **R** point p not on L.

The 2-Color Case

Thm For all $\mathrm{COL}\colon \mathbb{R}^2 \rightarrow [2]$ there is a mono triangle with area 1.
Case 1: $\exists$ a horiz. line L which is all **R**, and a **R** point p not on L.
Let q be point on L closest to p. $d = d(p, q)$:

The 2-Color Case

Thm For all $\mathrm{COL}\colon \mathbb{R}^2 \to [2]$ there is a mono triangle with area 1.
Case 1: $\exists$ a horiz. line L which is all **R**, and a **R** point p not on L.
Let q be point on L closest to p. $d = d(p, q)$:



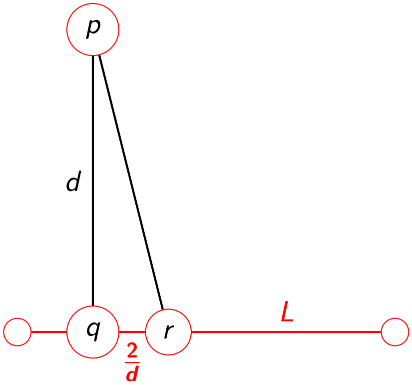
Red Triangle of Area 1

Red Triangle of Area 1

Let r be a point on L such that $d(q, r) = \frac{2}{d}$.

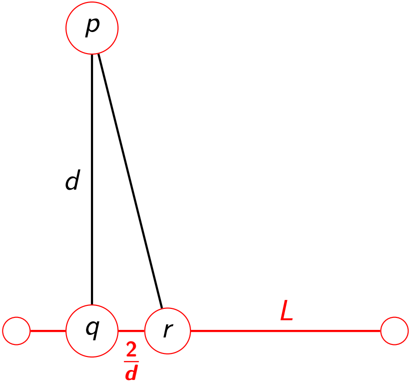
Red Triangle of Area 1

Let r be a point on L such that $d(q, r) = \frac{2}{d}$.



Red Triangle of Area 1

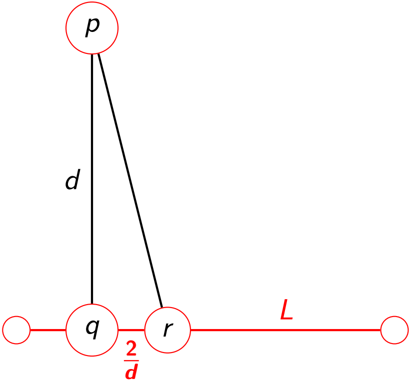
Let r be a point on L such that $d(q, r) = \frac{2}{d}$.



Area of triangle pqr is $\frac{1}{2} \times \frac{2}{d} \times d = 1$.

Red Triangle of Area 1

Let r be a point on L such that $d(q, r) = \frac{2}{d}$.



Area of triangle pqr is $\frac{1}{2} \times \frac{2}{d} \times d = 1$.
Case 1 DONE.

Case 2,3,4

The following cases are either trivial or similar to Case 1.

Case 2,3,4

The following cases are either trivial or similar to Case 1.
Case 2: $\exists$ a horiz. line L which is all **R**, but every p not on L is **B**.

Case 2,3,4

The following cases are either trivial or similar to Case 1.

Case 2: $\exists$ a horiz. line L which is all **R**, but every p not on L is **B**.

Case 3: $\exists$ a horiz. line L which is all **B**, and a **B** point p not on L.

Case 2,3,4

The following cases are either trivial or similar to Case 1.
Case 2: $\exists$ a horiz. line L which is all **R**, but every p not on L is **B**.

Case 3: $\exists$ a horiz. line L which is all **B**, and a **B** point p not on L.

Case 4: $\exists$ a horiz. line L which is all **B**, but every p not on L is **R**.

Case 2,3,4

The following cases are either trivial or similar to Case 1.

Case 2: $\exists$ a horiz. line L which is all **R**, but every p not on L is **B**.

Case 3: $\exists$ a horiz. line L which is all **B**, and a **B** point p not on L.

Case 4: $\exists$ a horiz. line L which is all **B**, but every p not on L is **R**.

So whats left? See next slide.

Case 5: Every Horiz Line has Both Colors

Case 5: Every Horiz Line has Both Colors

Case 5: Every horiz. line has both colors. We call this **mixed**.

Case 5: Every Horiz Line has Both Colors

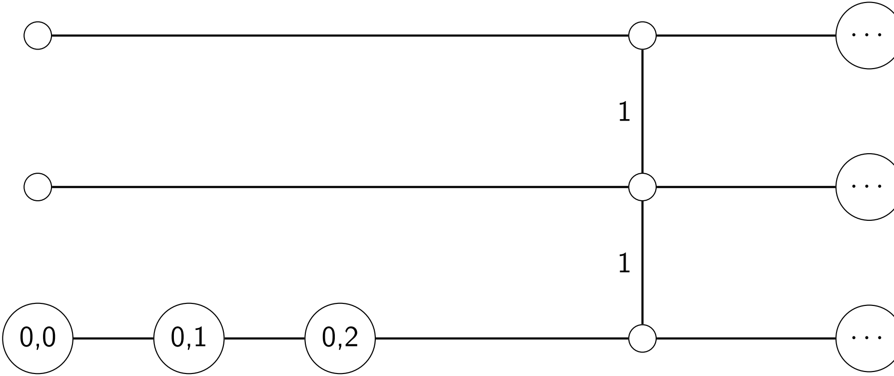
Case 5: Every horiz. line has both colors. We call this **mixed**.
We continue on next slide.

Three Key Points

We focus on $(0,0)$, $(0,1)$, $(0,2)$ and the infinite horiz. lines that are 1 and 2 above x-axis.

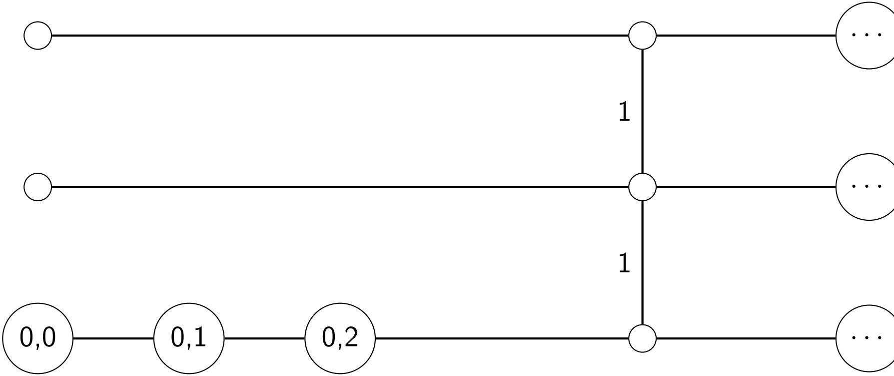
Three Key Points

We focus on $(0, 0)$, $(0, 1)$, $(0, 2)$ and the infinite horiz. lines that are 1 and 2 above x-axis.



Three Key Points

We focus on $(0,0)$, $(0,1)$, $(0,2)$ and the infinite horiz. lines that are 1 and 2 above x-axis.



Two of $(0,0)$, $(0,1)$, $(0,2)$ are the same color, say **R**.

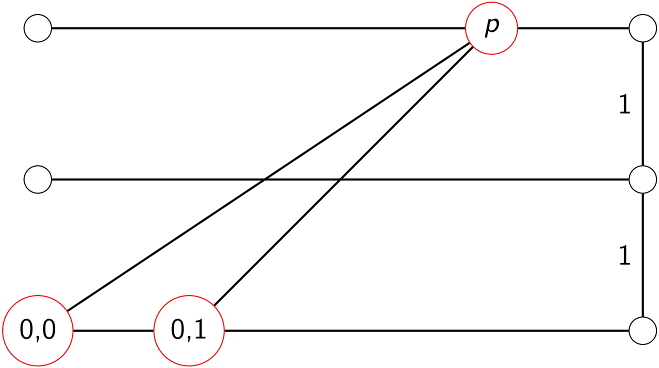
Case 5.1: $(0,0)$ and $(0,1)$ are R

Case 5.1: $(0,0)$ and $(0,1)$ are R

$\exists$ **R** p on top line since all horiz. lines are mixed.

Case 5.1: $(0,0)$ and $(0,1)$ are R

$\exists$ **R** p on top line since all horiz. lines are mixed.



Area of $(0,0), (0,1), p$ is $\frac{1}{2} \times 1 \times 2 = 1$.

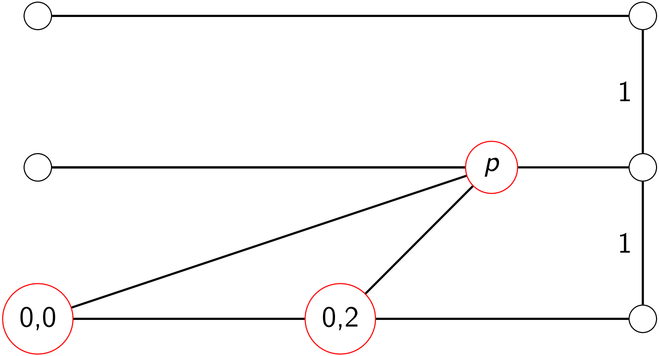
Case 5.2: $(0, 0)$ and $(0, 2)$ are R

Case 5.2: $(0, 0)$ and $(0, 2)$ are **R**

$\exists$ **R** p on middle line since all horiz. lines are mixed.

Case 5.2: $(0, 0)$ and $(0, 2)$ are R

$\exists$ **R** p on middle line since all horiz. lines are mixed.



Area of $(0, 0), (0, 2), p$ is $\frac{1}{2} \times 2 \times 1 = 1$.

The Three Color Case

The 3-Color Case

Thm For all $\mathrm{COL}\colon \mathbb{R}^2 \to [3]$ there is a mono triangle with area 1.

The 3-Color Case

Thm For all $\mathrm{COL}\colon \mathbb{R}^2 \to [3]$ there is a mono triangle with area 1.
We use the colors **R**, **B**, **G**.

The 3-Color Case

Thm For all $\mathrm{COL}\colon \mathbb{R}^2 \to [3]$ there is a mono triangle with area 1.
We use the colors **R**, **B**, **G**.
Thoughts

The 3-Color Case

Thm For all $\mathrm{COL}\colon \mathbb{R}^2 \to [3]$ there is a mono triangle with area 1.
We use the colors **R**, **B**, **G**.

Thoughts

1. The key to the 2-color case was that we had horiz. lines that all used **R** and **B**. We will try to get a set of horiz lines that all use **the same** colors.

The 3-Color Case

Thm For all $\mathrm{COL}\colon \mathbb{R}^2 \rightarrow [3]$ there is a mono triangle with area 1.
We use the colors **R**, **B**, **G**.

Thoughts

1. The key to the 2-color case was that we had horiz. lines that all used **R** and **B**. We will try to get a set of horiz lines that all use **the same** colors.
2. Another key is that the horiz. lines were equally spaced.

The 3-Color Case

Thm For all $\mathrm{COL}\colon \mathbb{R}^2 \to [3]$ there is a mono triangle with area 1.
We use the colors **R**, **B**, **G**.

Thoughts

1. The key to the 2-color case was that we had horiz. lines that all used **R** and **B**. We will try to get a set of horiz lines that all use **the same** colors.
2. Another key is that the horiz. lines were equally spaced.
3. So we need horiz. lines that all use the same set of colors and are equally spaced.

The 3-Color Case

Thm For all $\mathrm{COL}\colon \mathbb{R}^2 \rightarrow [3]$ there is a mono triangle with area 1.
We use the colors **R**, **B**, **G**.

Thoughts

1. The key to the 2-color case was that we had horiz. lines that all used **R** and **B**. We will try to get a set of horiz lines that all use **the same** colors.
2. Another key is that the horiz. lines were equally spaced.
3. So we need horiz. lines that all use the same set of colors and are equally spaced.
   What does this make you think of?

The 3-Color Case

Thm For all $\text{COL}\colon \mathbb{R}^2 \to [3]$ there is a mono triangle with area 1.
We use the colors **R**, **B**, **G**.

Thoughts

1. The key to the 2-color case was that we had horiz. lines that all used **R** and **B**. We will try to get a set of horiz lines that all use **the same** colors.
2. Another key is that the horiz. lines were equally spaced.
3. So we need horiz. lines that all use the same set of colors and are equally spaced.
   What does this make you think of?
   Answer on next slide.

Use VDW's Theorem

Use VDW's Theorem

Let $W = W(k, c)$ where we will pick k and c later.

Use VDW's Theorem

Let $W = W(k, c)$ where we will pick k and c later.

Define

$$\mathrm{COL}' \colon [W(k,c)] \rightarrow \{\{\mathbf{R}\}, \{\mathbf{B}\}, \{\mathbf{G}\}, \{\mathbf{R}, \mathbf{B}\}, \{\mathbf{R}, \mathbf{G}\}, \{\mathbf{B}, \mathbf{G}\}, \{\mathbf{R}, \mathbf{B}, \mathbf{G}\}\}$$

as follows:

Use VDW's Theorem

Let $W = W(k,c)$ where we will pick k and c later.

Define

$$\mathrm{COL}' \colon [W(k,c)] \to \{\{\mathbf{R}\},\{\mathbf{B}\},\{\mathbf{G}\},\{\mathbf{R},\mathbf{B}\},\{\mathbf{R},\mathbf{G}\},\{\mathbf{B},\mathbf{G}\},\{\mathbf{R},\mathbf{B},\mathbf{G}\}\}$$

as follows:
$\mathrm{COL}'(i) =$ the set of colors used by COL on the line $y = i$.

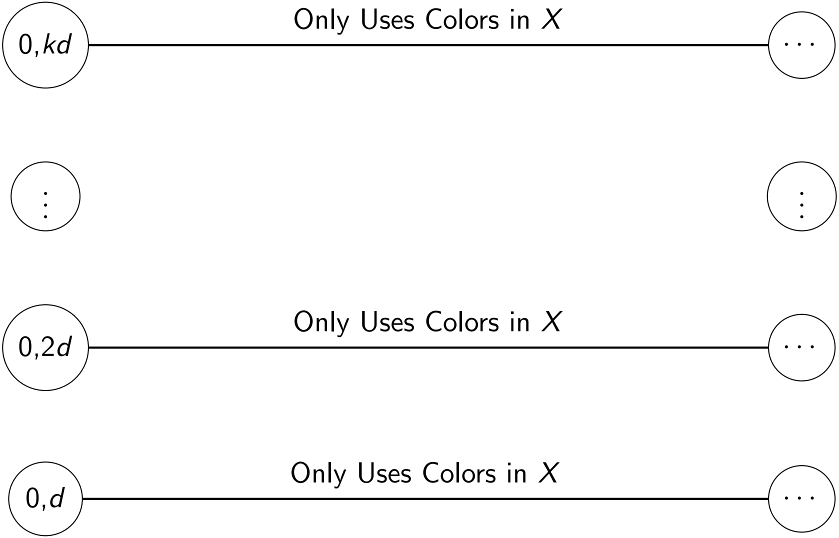
What Happens

What Happens

There exists $X \subseteq \{\mathbf{R}, \mathbf{B}, \mathbf{G}\}$ and d such that:

What Happens

There exists $X \subseteq \{\mathbf{R}, \mathbf{B}, \mathbf{G}\}$ and d such that:



Case 1: $|X| = 1$. Assume R

Case 1: $|X| = 1$. Assume **R**

Case 1: $|X| = 1$. Assume **R**

Area of $(0, d), (0, 2d), (\frac{2}{d}, d)$ is $\frac{1}{2} \times \frac{2}{d} \times d = 1$.

Case 2: $|X| = 2$. Assume $X = \{\mathbf{R}, \mathbf{B}\}$

Focus on $(0, d), (\frac{1}{3d}, d), (\frac{2}{3d}, d)$.

Case 2: $|X| = 2$. Assume $X = \{\mathbf{R}, \mathbf{B}\}$

Focus on $(0, d), (\frac{1}{3d}, d), (\frac{2}{3d}, d)$.

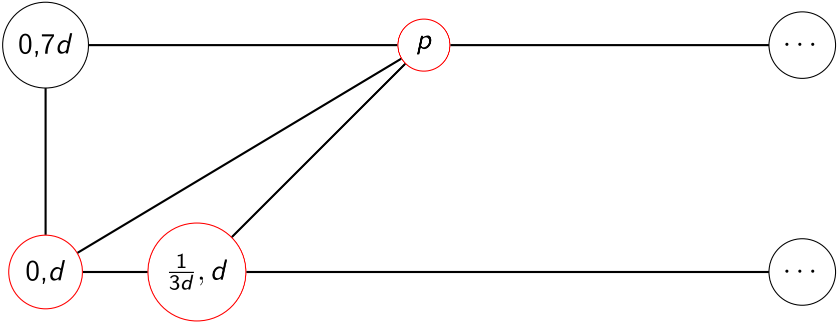
Two of them are the same color. Assume **R**.

Case 2.1: $|X| = 2$. $\mathbf{COL}(0, d) = \mathbf{COL}(\frac{1}{3d}, d) = \mathsf{R}$

Key Some point p on $6d$-horiz. line is **R**.

Case 2.1: $|X| = 2$. $\mathrm{COL}(0, d) = \mathrm{COL}(\frac{1}{3d}, d) = \mathbf{R}$

Key Some point p on $6d$-horiz. line is **R**.



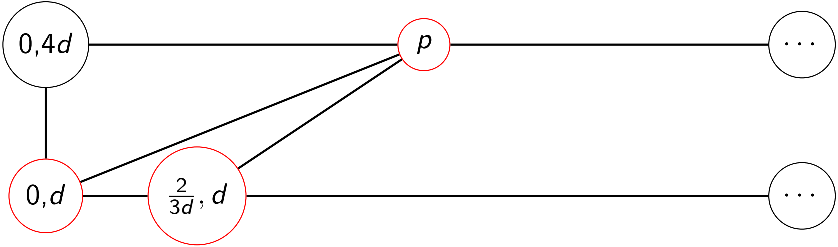
Area of triangle $((0, d), (\frac{1}{3d}, d), p)$ is $\frac{1}{2} \times \frac{1}{3d} \times 6d = 1$.

Case 2.2: $|X| = 2$. $\mathbf{COL}(0, d) = \mathbf{COL}(\frac{2}{3d}) =$ R

Key Some point p on $3d$-horiz. line is **R**.

Case 2.2: $|X| = 2$. $\mathrm{COL}(0, d) = \mathrm{COL}(\frac{2}{3d}) = \mathbf{R}$

Key Some point p on $3d$-horiz. line is **R**.



Area of triangle $((0, d), (\frac{2}{3d}, d), p)$ is $\frac{1}{2} \times \frac{2}{3d} \times 3d = 1$.

Case 3: $|X| = 3$

Case 3: $|X| = 3$

Focus on $(0, d), (\frac{1}{3d}, d), (\frac{2}{3d}, d), (\frac{1}{d}, d)$.

Case 3: $|X| = 3$

Focus on $(0, d), (\frac{1}{3d}, d), (\frac{2}{3d}, d), (\frac{1}{d}, d)$.

Key Two of $(0, d), (\frac{1}{3d}, d), (\frac{2}{3d}, d), (\frac{1}{d}, d)$ are same color.

Case 3: $|X| = 3$

Focus on $(0, d), (\frac{1}{3d}, d), (\frac{2}{3d}, d), (\frac{1}{d}, d)$.

Key Two of $(0, d), (\frac{1}{3d}, d), (\frac{2}{3d}, d), (\frac{1}{d}, d)$ are same color.

Old News If $\frac{1}{3d}$ apart or $\frac{2}{3d}$ apart then similar to Case 2.

Case 3: $|X| = 3$

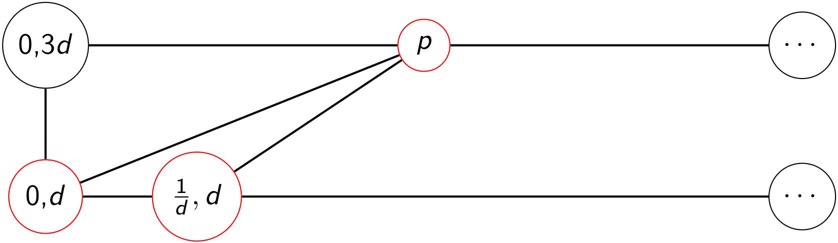
Focus on $(0, d), (\frac{1}{3d}, d), (\frac{2}{3d}, d), (\frac{1}{d}, d)$.

Key Two of $(0, d), (\frac{1}{3d}, d), (\frac{2}{3d}, d), (\frac{1}{d}, d)$ are same color.

Old News If $\frac{1}{3d}$ apart or $\frac{2}{3d}$ apart then similar to Case 2.

We Assume $\mathrm{COL}(0, d) = \mathrm{COL}(\frac{1}{d}, d) =$ **R**.

Case 3: $|X| = 3$



Area of triangle $((0, d), (\frac{1}{d}, d), p)$ is $\frac{1}{2} \times \frac{1}{d} \times 2d = 1$.

Fill in the Parameters

We used $W = W(k, c)$.

Fill in the Parameters

We used $W = W(k, c)$.

1. The colors are nonempty subsets of $\{\mathbf{R}, \mathbf{B}, \mathbf{G}\}$ so $c = 2^3 - 1 = 7$.

Fill in the Parameters

We used $W = W(k, c)$.

1. The colors are nonempty subsets of $\{\mathbf{R}, \mathbf{B}, \mathbf{G}\}$ so $c = 2^3 - 1 = 7$.
2. We need $7d$, so AP of length 7. $k = 7$.
3. **Upshot** Used $W(7, 7)$.

Generalize

Thm $(\forall c)(\forall \mathrm{COL}\colon \mathbb{R}^2 \to [c])$ $\exists$ mono triangle with area 1.

Generalize

Thm $(\forall c)(\forall \mathrm{COL}\colon \mathbb{R}^2 \rightarrow [c])$ $\exists$ mono triangle with area 1.

This is a HW problem.

Generalize

Thm $(\forall c)(\forall \mathrm{COL}\colon \mathbb{R}^2 \rightarrow [c])\ \exists$ mono triangle with area 1.

This is a HW problem.

Key is to find the right parameters for VDW.