A Survey of Anti-Ramsey Theorems
An Exposition by
William Gasarch and
Katherine Krotkov

1 Introduction

Definition 1.1 If $n \in \mathsf{N}$ then $[n] = \{1, \ldots, n\}$.

The following is Schur's Theorem which is an early result (1916) in Ramsey theory:

Theorem 1.2 *For all* $c \in \mathsf{N}$ *there exists* $n \in \mathsf{N}$ *such that, for all* $\mathrm{COL}\colon [n] \to [c]$ *there exists* x, y, z *all the same color such that* $x + y = z$.

In Schur's theorem the goal is a solution to $x + y = z$ where x, y, z are *the same* color. What if you want a solution where x, y, z are *different colors*?

Definition 1.3 Let $c, n \in \mathsf{N}$ and $\mathrm{COL}\colon [n] \to [c]$. A triple $x, y, z \in [n]$ is *rainbow* if x, y, z are all different colors.

If COL colors all the numbers R then there is no rainbow solution. Hence all of the theorems we discuss will have a lower bound on how often each color appears.

2 Each Color is Used on at least $1/3$ of the Numbers

The following theorem was proven in 1987 by V. Alekseev and S. Savchev. The paper is in Russian and is here:
`https://www.kvant.digital/problems/m1040/`

Theorem 2.1 *Let* $n \in \mathsf{N}$. *Let* $\mathrm{COL}\colon [3n] \to [3]$. *Assume that every color appears in the image* n *times. Then there exists rainbow* $x, y, z \in [3n]$ *such* $x + y = z$

Proof:
We let the colors be R,B,G. We can assume COL(1) = R. Let k be such that

$$\mathrm{COL}(1) = \cdots = \mathrm{COL}(k-1) \neq \mathrm{COL}(k).$$

We can assume COL(k) = B.

We do an example: $n = 7$, $k = 5$, and COL(13) = G. (When we generalize the example we will have a instead of 13). So we have:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
R	R	R	R	B								G								

We will show that either we get our rainbow solution or COL(12) = R. When we generalize this example (1) we will have $a - 1$ instead of 12, and (2) we will see how getting $\mathrm{COL}(a-1) = R$ helps prove the theorem.

Case 1: COL(12) = B. So we have:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
R	R	R	R	B							B	G								

Then $(1, 12, 13)$ is a rainbow solution. (In generalization, 12 is replaced with $a - 1$.)

Case 2: COL(12) = G. So we have:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
R	R	R	R	B							G	G								

Case 2.1: COL(8) = R. So we have:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
R	R	R	R	B			R				G	G								

Then $(5, 8, 13)$ is a rainbow solution. (This case did not use COL(12) = G.) (In generalization, 8 is replaced with $a - k$.)

Case 2.2: COL(8) = B. So we have:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
R	R	R	R	B			B				G	G								

Then $(4, 8, 12)$ is a rainbow solution.

Case 2.3: COL(8) = G. (We will go through cases that look odd; however, they are similar to what happens when we generalize this example.) So we have:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
R	R	R	R	B			G				G	G								

Case 2.3.1: COL(3) = B. NOT TRUE. (When we generalize this example we will have $a - 2k$ instead of 3.)

Case 2.3.2: COL(3) = R. TRUE. Rainbow solution $(3, 5, 8)$.

Case 2.3.2.1: COL(3) = G. NOT TRUE.

Case 3: COL(12) = R. This must be what happens.

We generalize this example.

Case 1: $\exists a \geq k + 1$, COL(a) = G and COL($a - 1$) = B. Then $(1, a - 1, a)$ is a rainbow solution.

Case 2: $\exists a \geq k + 1$, COL(a) = G and COL($a - 1$) = G.

Case 2.1: COL($a - k$) = R. ($a - k \in [n]$ since $a \geq k + 1$.)

Then $(k, a - k, a)$ is a rainbow solution. (This case did not use that COL($a - 1$) = G.)

Case 2.2: COL($a - k$) = B.

Then $(k - 1, a - k, a - 1)$ is a rainbow solution.

Case 2.3: COL($a - k$) = G. Replicate the reasoning from Case 1,2,2.1,2.2 with a replaced by $a - k$. There is either a rainbow solution or COL($a - 2k$) = G. If COL($a - 2k$) = G then (1) $a - 2k \in [k]$: contradiction, or (2) repeat the argument again. Keep doing this. Eventually there is a rainbow solution since otherwise $\exists i$, $a - ik \in [k]$ and hence cannot be green.

Case 3: $\forall a$ if COL(a) = G then COL($a - 1$) = R. We map every green number a to $a - 1$. This is an injection and everything in the image is red. The number 1 is not in the image since either COL(2) = R or COL(2) = B. Hence there is at least one more green number than red number. But the number of green numbers is the same as the number of red numbers (both numbers are n). Hence this case cannot occur.

∎

3 Each Color is Used on at least $1/4$ of the Numbers: No Theorem

Recall Theorem 2.1:

Theorem 3.1 *Let $n \in \mathsf{N}$. Let* $\mathrm{COL}\colon [3n] \to [3]$. *Assume that every color appears in the image n times. Then there exists rainbow $x, y, z \in [3n]$ such $x + y = z$*

We give a definition for convenience and restate Theorem 2.1.

Definition 3.2 Let $m, c \in \mathsf{N}$. Let $\mathrm{COL}\colon [m] \to [c]$. A *rainbow solution* is x, y, z such that (1) they are all different colors, and (2) $x + y = z$.

We restate Theorem 2.1.

Theorem 3.3 *Let $n \in \mathsf{N}$ such that $n \equiv 0 \pmod 3$. Let* $\mathrm{COL}\colon [n] \to [3]$. *Assume that every color appears in the image $\frac{n}{3}$ times. Then there exists a rainbow solution.*

In Theorem 3.3 we demanded that R, B, G appear as much as possible. Can this be improved? What if we demanded that R, B, G each appear $\geq \frac{n}{4}$ times? Alas, then the theorem fails. The following is folklore. We include the proof for completeness.

Theorem 3.4 *Let $n \in \mathsf{N}$ such that $n \equiv 0 \pmod 4$. There exists* $\mathrm{COL}\colon [n] \to [3]$ *such that (1) every color appears $\geq \frac{n}{4}$ times, and (2) there is no rainbow solution.*

Proof:

$$\mathrm{COL}(w) = \begin{cases} \mathrm{R} & \text{if } 1 \leq w \leq \frac{n}{2} \text{ and } w \equiv 0 \pmod 2 \\ \mathrm{B} & \text{if } w \equiv 1 \pmod 2 \\ \mathrm{G} & \text{if } \frac{n}{2} + 1 \leq w \leq n \text{ and } w \equiv 0 \pmod 2 \end{cases} \tag{1}$$

There are $\frac{n}{4}$ red numbers, $\frac{n}{2}$ blue numbers, and $\frac{n}{4}$ green numbers.

If x, y, z is a rainbow solution then $x + y = z$ where they are different colors. Since all red numbers are even, all blue numbers are even, and all green numbers are odd, this is impossible. ∎

Is there some fraction $\frac{1}{4} < \alpha < \frac{1}{3}$ such that the following holds?

Let $n \in \mathsf{N}$. Let $\mathrm{COL}\colon [n] \to [3]$. *Assume that every color appears in the image $\geq \alpha n$ times. Then there exists rainbow solution.*

Actually any $\alpha > \frac{1}{4}$ works. In fact, the situation is better than that. Our main theorem states that if all colors appear $\geq \frac{n}{4} + 1$ times then there is a rainbow solution.

4 Each Color is Used on More Than $1/4$ of the Numbers

Esther and George Szekeres [2] proved the following.

Theorem 4.1 *Let* $\mathrm{COL}\colon [n] \to [3]$ *be such that R,B,G all appear $> \frac{n}{4}$ times. Then there exists a rainbow solution.*

Proof:

Let COL: $[n] \rightarrow [3]$ be such that R,B,G all appear $> n/4$ times.

Let

- $R = \{x_1 < \cdots < x_a\}$ be the set of RED elements,
- $B = \{y_1 < \cdots < y_b\}$ be the set of BLUE elements,
- $G = \{z_1 < \cdots < z_c\}$ be the set of GREEN elements,

We will say $w \in R$ instead of $\mathrm{COL}(w) = R$.

We can assume $x_1 = 1$ and $x_1 < y_1 < z_1$.

Note the following.

Note 4.2

1. If $w < y_1$ then $w \in R$.
2. If $w < z_1$ then $w \in R \cup B$.
3. $y_1 \geq 2$.
4. $z_1 \geq 4$ (If $z_1 = 2$ then $1 < y_1 < 2$. If $z_1 = 3$ then $y_1 = 2$ and x_1, y_1, z_1 is a rainbow solution.)

Notation 4.3

1. A pair of elements are *adjacent elements of* G if they are of the form (z_i, z_{i+1}).
2. Let r be the minimum difference between adjacent elements of G. Formally

$$r = \min_{1 \leq k \leq c-1} z_{k+1} - z_k.$$

3. Let k be the least k such that $z_{k+1} - z_k = r$.

Informally, we are interested in r since, if r is large, then G is small, perhaps $\leq n/4$.

There are five cases. Each case assumes the negation of the prior cases. The cases will have subcases. Every case and subcase concludes either (1) there is a rainbow solution, or (2) the case cannot occur.

Begin The Five Cases

Case 1 $r \geq 4$. By Note 4.2.4, $z_1 \geq 4$. Since $r \geq 4$, inductively, $(\forall 1 \leq i \leq c)[z_i \geq 4i]$. Hence $4c \leq z_c \leq n$, so $c \leq \frac{n}{4}$. This is contrary to hypothesis so cannot occur.

Case 2 $r < y_1$. By Note 4.2.1, $r \in R$ and $y_1 - r \in R$.

We look at the color of $z_k + r - y_1$ (since $r < y_1$, $z_k + r - y_1 < z_k$). First we need that $z_k + r - y_1 \geq 1$.

If $z_k + r - y_1 \leq 0$ then $z_k \leq y_1 - r \leq y_1$, so by Note 4.2.1 $z_k \in R$, but we know that $z_k \in G$. So $z_k + r - y_1 \geq 1$.

Color of $z_k + r - y_1$

1. $z_k + r - y_1 \in R$. Then

$$\underset{R}{(z_k + r - y_1)} + \underset{B}{y_1} = \underset{G}{z_k + r} = z_{k+1}$$

is a rainbow solution.

2. $z_k + r - y_1 \in B$. Then

$$\underset{B}{(z_k + r - y_1)} + \underset{R}{(y_1 - r)} = \underset{G}{z_k}$$

is a rainbow solution.

3. $z_k + r - y_1 \in G$. Then there exists k' such that $z_k + r - y_1 = z_{k'}$. Since $z_k + r - y_1 < z_k$, $k' < k$. We will use this later.

   For now, we look at $z_k - y_1$.

   (a) $z_k - y_1 \in R$. Then

   $$\underset{R}{(z_k - y_1)} + \underset{B}{y_1} = \underset{G}{z_k}$$

   is a rainbow solution.

   (b) $z_k - y_1 \in B$. Then

   $$\underset{B}{(z_k - y_1)} + \underset{R}{r} = \underset{G}{z_k + r - y_1}$$

   is a rainbow solution.

   (c) $z_k - y_1 \in G$. We are not going to get a rainbow solution. We will contradict the definition of k.
   If $z_k - y_1 \in G$ then there exists k'' such that $z_k - y_1 = z_{k''}$. Since $z_k - y_1 < z_k + r - y_1$, $k'' < k'$.
   $z_{k'} - z_{k''} = z_k + r - y_1 - (z_k - y_1) = r$
   Recall that k is the *least* number such that $z_{k+1} - z_k = r$. But we have just seen that $z_{k'} - z_{k''} = r$ and $k'' < k$. Hence this subcase cannot occur.

Recap We assume the negation of Case 1 and Case 2 which we denote $\neg C1$ and $\neg C2$.

1. $\neg C1$ implies $r \le 3$.

2. $\neg C2$ and Note 4.2.3 implies $2 \le y_1 \le r$.

By the two points above $r \in \{2, 3\}$. By $\neg C2$, $y_1 \le r$. Hence

$$(r, y_1) \in \{(3,3), (3,2), (2,2)\}.$$

Therefore there are three more cases.

Case 3 $(r, y_1) = (3,3)$. Since $y_1 = 3$, by Note 4.2.1 we have $1, 2 \in R$.

We consider the colors of $z_t - 1$ and $z_t - 2$ for all $1 \le t \le c$.

Color of $z_t - 1$

- If $z_t - 1 \in B$ then
$$\begin{array}{ccc} (z_t-1)+ & 1 & = z_t \\ B & R & G \end{array}$$
is a rainbow solution.

- If $z_t - 1 \in G$ then $z_t - 1 = z_{t-1}$ and z_t are adjacent elements of G with difference 1. This contradicts the minimum difference between two adjacent elements of G being $r = 3$. Hence this subcase cannot occur.

- We can assume $(\forall t)[z_t - 1 \in R]$.

Color of $z_t - 2$

- If $z_t - 2 \in B$ then
$$\begin{array}{ccc} (z_t-2)+ & 2 & = z_t \\ B & R & G \end{array}$$
is a rainbow solution.

- If $z_t - 2 \in G$ then $z_t - 2 = z_{t-1}$ and z_t are adjacent elements of G with difference 2. This contradicts the minimum difference between two adjacent elements of G being $r = 3$. Hence this subcase cannot occur.

- We can assume $(\forall t)[z_t - 2 \in R]$.

To recap: for all $z \in G$, $z - 1, z - 2 \in R$. Hence
$a \ge 2c$
Since $c > \frac{n}{4}$ we get $a > \frac{n}{2}$.
Since $a + b + c = n$

$$b = n - a - c < n - \frac{n}{2} - \frac{n}{4} = \frac{n}{4}$$

This is contrary to hypothesis.

Case 4 $(r, y_1) = (3, 2)$. So $1 \in R$ and $2 \in B$.
Note that $z_{k+1} = z_k + 3$.
We look at the color of $z_k + 1$.

Color of $z_k + 1$

- If $z_k + 1 \in R$ then
$$\begin{array}{ccc} (z_k+1) & +2 & = z_k + 3 = z_{k+1} \\ R & B & G \end{array}$$
is a rainbow solution.

- If $z_k + 1 \in B$ then
$$\begin{array}{ccc} z_k & +1 & = z_k + 1 \\ G & R & B \end{array}$$
is a rainbow solution.

- If $z_k+1 \in G$ then z_k+1 and z_k are adjacent elements of G with difference 1. This contradicts that minimum different between adjacent elements of G is $r=3$. Hence this subcase cannot occur.

Case 5 $(r, y_1) = (2, 2)$. So $1 \in R$ and $2 \in B$.

We will look at the colors of $z_k - 1$, $z_k + 1$, $z_k - 2$, and 3.

Color of $z_k - 1$

- If $z_k - 1 \in B$ then

$$\begin{array}{ccc} (z_k - 1) & +1 & = z_k \\ B & R & G \end{array}$$

  is a rainbow solution.

- If $z_k - 1 \in G$ then $z_{k-1} = z_k - 1$ and z_k are adjacent elements of G that are 1 apart, which contradicts $r = 2$. Hence this subcase cannot occur.

- We can assume $z_k - 1 \in R$.

Color of $z_k + 1$

- If $z_k + 1 \in B$ then

$$\begin{array}{ccc} z_k & +1 & = (z_k + 1) \\ G & R & B \end{array}$$

  is a rainbow solution.

- If $z_k + 1 \in G$ then $z_{k+1} = z_k + 1$ and z_k are two adjacent elements of G that are 1 apart, which contradicts $r = 2$. Hence this cannot occur.

- We can assume $z_k + 1 \in R$.

Color of $z_k - 2$

- If $z_k - 2 \in R$ then

$$\begin{array}{ccc} (z_k - 2) & +2 & = z_k \\ R & B & G \end{array}$$

  is a rainbow solution.

- If $z_k - 2 \in G$ then note that $z_k - 2, z_k - 1, z_k$ are three consecutive numbers and that (from the case Color of $z_k - 1$) we know that $z_k - 1 \in R$. Hence within G we have $z_k - 2, z_k$ are adjacent, so $z_k - 2 = z_{k-1}$. This contradicts that k is the least index with $z_{k+1} - z_k = r = 2$. Hence this subcase cannot occur.

- We can assume $z_k - 2 \in B$.

Color of 3

- If $3 \in B$ then

$$\begin{array}{ccc} (z_k - 1) & +3 & = z_k + 2 = z_{k+1} \\ R & B & G \end{array}$$

  is a rainbow solution.

- If $3 \in G$ then

$$\begin{array}{ccc} 1 & +2 & = 3 \\ R & B & G \end{array}$$

  is a rainbow solution.

- We can assume $3 \in R$.

Claim There is a rainbow solution or every odd number $\leq n$ is in R.
Proof

We try to prove that every odd number $\leq n$ is in R. We will either succeed or show there is a rainbow solution. The proof is by induction.

Base Case Since we are in Case 5, $1 \in R$. We have also shown $3 \in R$.
Induction Hypothesis $i \geq 1$ and $2i-1, 2i+1 \in R$.
Induction Step We show $2i+3 \in R$.

We consider two cases: $2i+1 < z_k$ and $z_k < 2i+1$. Note that $2i+1 = z_k$ is impossible since $2i+1 \in R$ and $z_k \in G$.

Case 5.1 $2i+1 < z_k$.
Color of $z_k - (2i+1)$ (This is ≥ 1 by the Case we are in.)

- If $z_k - (2i+1) \in B$ then

$$\begin{array}{ccc} (2i+1) & +(z_k - (2i+1)) & = z_k \\ R & B & G \end{array}$$

  which is a rainbow solution.

- If $z_k - (2i+1) \in G$ then

$$\begin{array}{ccc} (z_k - (2i+1)) & +(2i-1) = & z_k - 2 \\ G & R & B \end{array}$$

  which is a rainbow solution

- We can assume $z_k - (2i+1) \in R$

Color of $2i+3$

- If $2i+3 \in B$ then

$$\begin{array}{ccc} (z_k - (2i+1)) & +(2i+3) & = z_k + 2 = z_{k+1} \\ R & B & G \end{array}$$

  which is a rainbow solution

- If $2i+3 \in G$ then

$$\begin{array}{ccc} (2i+1) & +2 & =2i+3 \\ R & B & G \end{array}$$

which is a rainbow solution

- We have $2i+3 \in$ R which was our goal.

Case 5.2 $z_k < 2i+1$.
Color of $2i+1-z_k$ (This is ≥ 1 by the Case we are in.)

- If $2i+1-z_k \in B$ then

$$\begin{array}{ccc} z_k & +(2i+1-z_k) & =2i+1 \\ G & B & R \end{array}$$

is a rainbow solution.

- If $2i+1-z_k \in G$ then

$$\begin{array}{ccc} (z_k-2) & +(2i+1-z_k) & =2i-1 \\ B & G & R \end{array}$$

is a rainbow solution.

- We can assume $2i+1-z_k \in R$.

Color of $2i+3$

- If $2i+3 \in B$ then

$$\begin{array}{ccc} (z_k+2) & +(2i+1-z_k) & =(2i+3) \\ G & R & B \end{array}$$

which is a rainbow solution

- If $2i+3 \in G$ then

$$\begin{array}{ccc} (2i+1) & +2 & =2i+3 \\ R & B & G \end{array}$$

- We have $2i+3 \in R$ which was or goal.

End of Proof

Since every odd number $\leq n$ is in R, $|R| \geq n/2$. Hence $|B|+|G| \leq n/2$, so either $|B| \leq n/4$ or $|G| \leq n/4$. Either one contradicts the premise of the theorem. Hence this subcase cannot occur.
End of the Five Cases

Recap: There were 5 cases and several subcases. Every single case either (1) yields a rainbow solution, or (2) cannot occur. Hence there is a rainbow solution. ∎

5 What Else is Known?

Theorem 2.1 raises some questions:

What if we have a different equation? Fox, Mahdian, and Radoicic [1] showed the following: *For all n, for all* COL: $[n] \rightarrow [4]$ *where every color appears* $\geq \frac{n+1}{6}$ *times, there is a rainbow solution to $x+y=z+w$. The lower bound on the number of times a color can appear is tight.*

We are sure that other equations have been studied. We will add those later.

References

[1] Jacob Fox, Mohammad Mahdian, and Rados Radoicic. Raibnow solutio to the Sidon equation. *Discrete Mathematics*, 308:4773–4778, 2008.
https://mfeapp.baruch.cuny.edu/math/wp-content/uploads/2015/10/20.pdf.

[2] Esther Szekeres and George Szekeres. Adding numbers. *James Cook Math Notes*, 35:4073–4075, 1984.
https://webhomes.maths.ed.ac.uk/cook/iss_no35.PDF.