Bill Gasarch's Ramsey class, 4/22/2025; lecture by Aravind Srinivasan

Today: both lower and upper bounds on Ramsey numbers

Recall Ramsey R(s,t) [where we use two colors and only work with graphs, not hypergraphs]: the smallest n such that in any two-coloring of the edges of K_n, there is either a red clique with s vertices or a blue clique with t vertices (alternatively, a clique with s vertices or an independent set with t vertices)

We will start with the case s << t (say s = O(1)): Ramsey bound R(s,t) <= ((s + t - 2) choose (s -1)) suggests, for fixed s, that R(s,t) could be of the form t^{s - 1 - o(1)}; will first study this and progress on lower bounds for van der Waerden through the LLL

Lower Bounds

LLL, symmetric version: show the lemma and also discuss the algorithmic aspect

- Small improvements on lower bounds for R(t, t)
- Hypergraph coloring and [Radhakrishnan-Srinivasan]'s improvement; [RS] idea **(open problem: improve!)**
- Version with lower-bounded edge-size: Duraj-Gutowski-Kozic **(open problem: improve!)** [didn't have time for this]
- Van der Warden:
    - Szemeredi's theorem
    - Lower bounds on W(k) = W_2(k) via the LLL; Berlekamp; Shelah's upper bound
    - Kelley-Meka: there is a (small-ish) constant beta > 0 such that any subset S of [N] with size at least N * exp(-(log N)^{beta}) contains a non-trivial AP! Behrend's construction in the other direction for infinitely many N, with beta = ½

LLL, asymmetric version; algorithmic aspect works here as well [didn't have time for this]

- Upper bound on R(3,t): O(t^2) easy, O(t^2 / log t) [Ajtai-Komlos-Szemeredi, Shearer]
- Lower bounds on R(3,t) to get O(t^2 / log^2 t) [Erdos—complicated approach] and via random graphs and the asymmetric LLL: proof idea shown now in class (**random constructions:** a powerful way to show the existence of various combinatorial structures)
- Optimal Omega(t^2 / log t) by Kim: random graphs, Lovasz Local Lemma, the "nibble method" (also, proof via differential equations by Spencer)

R(4,t): near-optimal lower bound of Omega(t^3 / log^ 4 t) due to Mattheus-Verstraete, using *unitals in finite geometry*; **want to take on R(5,t)?**

Upper bound on R(t,t): survey by Yuval Wigderson

Recall $R(s,t)$

$R(t,t) \le 4^t$ (standard), $\le (4 - o(1))^t$

Campos-Griffiths-Morris-Sahasrabudhe: $R(t,t) \le 3.993^t$ – a breakthrough: current-best $\le 3.8^t$

Erdos-Szekeres argument that shows $R(s,t) \le R(s-1, t) + R(s, t-1)$

Book graph $B(m, p)$: [note: Wigderson uses the notation $B(t,m)$] **spine** A that's a clique with m vertices; **pages** Y that are p vertices, each adjacent to all of A (consider the special cases where $p = 1$ and $m = 1$ for intuition); **red books, blue books.**

Easy Theorem: Suppose a two-coloring of the edges of K_n contains a **red** book $B(m,p)$, where $p \ge R(s - m, t)$. Then this coloring contains a red K_s or a blue K_t.

Proof: Simple.

Approach: "book algorithm" that will find a good book

Erdos-Szekeres argument as a book algorithm:

Maintain three disjoint sets of vertices A, B, X; A, B start at the empty set and grow, while X will shrink; also, **(A,X) always a red book, (B,X) always a blue book**

Algorithm:
Suppose for a contradiction that for K_N with $N = 4^t$, there is a coloring with no blue or red K_t.

- If $|X| \le 1$, $|A| \ge t$, or $|B| \ge t$, **halt; (note: done** if the second or third conditions hold)
- Pick an arbitrary v in X; if it has at least $(|X| - 1)/2$ red neighbors **in X,** then move v to A and shrink X to the red neighborhood of v; similarly for the remaining case.
- **(Note that the invariants are maintained; also, we always make progress)**

Key Claim: $|X| \ge 2^{-|A| + |B|} * N$ always; prove by induction on time with the (carefully justifiable) approximation "$(|X| - 1)/2$ is approximately $|X| / 2$".

Thus, at the end, $1 \ge 2^{-2(t-1)} N$, a contradiction.

Similarly for off-diagonal Ramsey, where the "½" in the algorithm's "$(|X| - 1)/2$" is replaced by a suitable fraction gamma.

[CGMS]: A more-sophisticated book-finding algorithm: now four disjoint sets A, B, X, Y where the above invariant holds, **and** that (A,Y) is a red book (either these, or the case with red and blue interchanged)

Paul Balister, Béla Bollobás, Marcelo Campos, Simon Griffiths, Eoin Hurley, Robert Morris, Julian Sahasrabudhe, and Marius Tiba (2024): new approach via self-correlations of high-dimensional distributions

Also see, related to this class:

Yuval Wigderson (2024). “Ramsey theory—lecture notes”. url: https://n.ethz.ch/~ywigderson/math/static/RamseyTheory2024LectureNotes.pdf.