On Collections of Sets of Size 5 Where All Pairs have Intersection of Size 1
by William Gasarch

1 A Question From the 2025 UMCP HS Math Competition

We paraphrase a question from the 2025 University of Maryland High School Mathematics Competition.

Notation 1.1 If A is a finite set then $|A|$ is the number of elements in A.

Convention 1.2 We refer to finite sets as *blocks* throughout.

Definition 1.3 A collection of blocks $A_1, \ldots, A_m \subseteq \{1, \ldots, 2025\}$ is called *awesome* if

- for all $1 \le i \le m$, $|A_i| = 5$, and
- for all $1 \le i < j \le m$, $|A_i \cap A_j| = 1$.
- for all $1 \le y \le 2025$, y is in at least one of the A_i's.

The Problem

Find the natural number m such that the following are both true

- *There exists an awesome collection of* m *blocks.*
- *There does not exist an awesome collection of* $m+1$ *blocks.*

The answer will follow easily from a general theorem that we prove in the next section.

2 The General Problem

Definition 2.1 Let $n \in \mathsf{N}$. A collection of blocks $A_1, \ldots, A_m \subseteq \{1, \ldots, n\}$ is called n*-awesome* if the following hold:

- For all $1 \le i \le m$, $|A_i| = 5$.
- For all $1 \le i < j \le m$, $|A_i \cap A_j| = 1$.
- For all $1 \le y \le n$, y is in at least one of the A_i's.

Notation 2.2 Let $f(n)$ be the maximum value of m such that there is an n-awesome collection of size m if such an m exists. If no such m exists then we define $f(n)$ to be $*$.

3 This is Not Combinatorial Designs

Combinatorial design theory deals with collections of sets that satisfy certain properties. Our problem does *is not* one of those that is studied. We give examples of what is studied to note the difference.

Definition 3.1 A *balanced incomplete block design (BIBD)* with parameters (n, k, λ) is a collection $A_1, \ldots, A_m$ of $\{1, \ldots, n\}$ such that the following hold.

1. For all $1 \le i \le m$, $|A_i| = k$.
2. For all $1 \le y \le n$, y appears in r of the A_i's.
3. For all $1 \le y_1 < y_2 \le n$, y_1 and y_2 appear in λ of the A_i's.

Note 3.2 Our problem is not about BIBD's since (a) BIBD's do not involve intersection, (b) BIBD's require that every element appears in the same number of blocks, and (c) BIBD's have a condition on pairs of elements.

4 $f(n)$ for $n \ge 106$

Theorem 4.1 *If* $n \equiv 1 \pmod 4$ *then* $f(n) \ge \frac{n-1}{4}$.

Proof: Let the collection be

$$\{1, 2, 3, 4, n\}, \{5, 6, 7, 8, n\}, \ldots, \{n-4, n-3, n-2, n-1, n\}.$$

The number of blocks is $\frac{n-1}{4}$. ∎

Lemma 4.2 *Let* $n \in \mathsf{N}$ *and* $A_1, \ldots, A_m$ *be an* n*-awesome collection. Assume that there exists* $y \in [n]$ *that is in* ≥ 6 *of the* A_i*'s. Then:*

1. y *is in every* A_i.
2. $n \equiv 1 \pmod 4$.
3. $m = \frac{n-1}{4}$.

Proof:
1) By renumbering we can assume that the blocks that have y are
$\{1, 2, 3, 4, y\}$
$\{5, 6, 7, 8, y\}$
$\{9, 10, 11, 12, y\}$
$\{13, 14, 15, 16, y\}$
$\{17, 18, 19, 20, y\}$
$\{21, 22, 23, 24, y\}$

Assume, by way of contradiction, that there is block A that does not have y. It must have a non-y element from each of the six blocks above. Those non-y elements are all different. Hence $|A| = 6$, which is a contradiction.

2 and 3) The collection is a partition of $[n] - \{y\}$ into m sets of size 4. Hence $n = 4m + 1$, so

- $n \equiv 1 \pmod 4$ and
- $m = \frac{n-1}{4}$.

▮

Lemma 4.3 *Let $n \not\equiv 1 \pmod 4$. Let $n \in \mathsf{N}$ and $A_1, \ldots, A_m$ be an n-awesome collection. Then, for all $y \in [n]$, y appears in ≤ 5 blocks.*

Proof: This follows from Lemma 4.2. ▮

Lemma 4.4 *Let $n \in \mathsf{N}$ and $A_1, \ldots, A_m$ be an n-awesome collection. Assume that, for all $y \in [n]$, y appears in ≤ 5 of the A_i's. Then $n \le 105$.*

Proof:

Let the blocks be $A_1, \ldots, A_m$. By renumbering we can assume $A_1 = \{1, 2, 3, 4, 5\}$. Map each $2 \le i \le m$ to the unique number in $\{1, 2, 3, 4, 5\}$ that is in A_i. By the premise, the map is $\le$ 5-to-1. Hence $m \le 25$.

Let $I_1, \ldots, I_5$ be such that, for all $i \in I_j$, A_i maps to j.
$\bigcup_{j \in I_1} A_i$ has $\le 4|I_1| + 1$ elements.
$\bigcup_{j \in I_2} A_i$ has $\le 4|I_2| + 1$ elements.
$\bigcup_{j \in I_3} A_i$ has $\le 4|I_3| + 1$ elements.
$\bigcup_{j \in I_4} A_i$ has $\le 4|I_4| + 1$ elements.
$\bigcup_{j \in I_5} A_i$ has $\le 4|I_5| + 1$ elements.
Hence

$$n \le 4(|I_1| + \cdots + |I_5|) + 5 \le 4m + 5 \le 4 \times 25 + 5 = 105.$$

▮

Theorem 4.5

1. *If $n \equiv 1 \pmod 4$ and $n \ge 106$ then $f(n) = \frac{n-1}{4}$.*
2. *If $n \not\equiv 1 \pmod 4$ and $n \ge 106$ then $f(n) = *$.*

Proof:

1) By Theorem 4.1, $f(n) \ge \frac{n-1}{4}$. Let $A_1, \ldots, A_m$ be an n-awesome collection.

- If there exists $y \in [n]$ that is in ≥ 6 of the A_i's, then, by Lemma 4.2, $m = \frac{n-1}{4}$.

- If for all $y \in [n]$, y appears in ≤ 5 of the A_i's, then, by Lemma 4.4, $n \leq 105$. This cannot happen since $n \geq 106$.

2) Let $A_1, \ldots, A_m$ be an n-awesome collection. By Lemma 4.3, for all $y \in [n]$, y appears in ≤ 5 of the A_i's. By Lemma 4.4, $n \leq 105$. ∎

Note 4.6 We can now solve the UMCP high school math competition problem. By Theorem 4.5 $f(2025) = 506$. Note that the proof is appropriate for a High School Math competition.

5 Lemmas needed for $f(n)$ for $n \geq 34$

Definition 5.1 Let $A_1, \ldots, A_m$ be an n-awesome collection of blocks. An *incidence* is an ordered pair $((y, i) \in [n] \times [m]$ such $y \in A_i$.

Notation 5.2 Assume $A_1, \ldots, A_m$ is an n-awesome collection of blocks.

1. For $1 \leq i \leq m$ let n_i be the number of elements of $\{1, \ldots, n\}$ appearing in exactly i blocks.
2. For $1 \leq i \leq n$ let r_i be the number of blocks that i is in.

We do an example that will lead to three linear equations in the n_i's.

Example 5.3 We look at $n = 14$, $m = 5$ and the blocks are:
$\{1, 2, 3, 4, 9\}$
$\{5, 6, 7, 8, 9\}$
$\{1, 5, 10, 11, 12\}$
$\{3, 7, 10, 13, 14\}$

1. $n_1 = 8$ since there are 8 elements that appear in exactly one block : $\{2, 4, 6, 8, 11, 12, 13, 14\}$.

   $n_2 = 6$ since there are 6 elements that appear in exactly two blocks: $\{1, 3, 5, 7, 9, 10\}$.

   $n_3 = n_4 = n_5 = 0$ since there are 0 elements that appear exactly three or four of five blocks.

   Note that the $\{2, 4, 6, 8, 11, 12, 13, 14\}$ and $\{1, 3, 5, 7, 9, 10\}$ form a partition of $[14]$.

   More generally, there will always be a partition of $[n]$ into $\leq m$ blocks based on the n_i. We generalize this and prove it later.

2. How many incidences are there? One answer is of course $4 \times 5 = 20$.

   Lets view it another way.

   $n_1 = 8$ via $\{2, 4, 6, 8, 11, 12, 13, 14\}$. Each of these contributes 1 to the number of incidences, so the entire contribution is $1 \times n_1 = 8$.

   $n_2 = 6$ via $\{1, 3, 5, 7, 9, 10\}$. Each of these contributes 2 to the number of incidences., so the entire contribution is $2 \times n_2 = 12$.

   $n_3 = n_4 = n_5 = 0$. Even though its 0, note that these contribute $3n_3 + 4n_4 + 5n_5 = 0$.

   Hence $1 \times n_1 + 2 \times n_2 + 3 \times n_3 + 4 \times n_4 + 5 \times n_5 = 4 \times 5$. We generalize this and prove it later.

3. We list out all of the r_i's:.

   $r_1 = 2$

   $r_2 = 1$

   $r_3 = 2$

   $r_4 = 1$

   $r_5 = 2$

   $r_6 = 1$

   $r_7 = 2$

   $r_8 = 1$

   $r_9 = 2 = r_{10} = 2$

   $r_{11} = r_{12} = r_{13} = r_{14} = 1$

   Note that $S = \sum_{i\in[n]} r_i^2 = 32$.

   We now view S in a different way.

   $n_1 = 8$ via $\{2, 4, 6, 8, 11, 12, 13, 14\}$. Each of these contributes 1 to S.

   $n_2 = 6$ via $\{1, 3, 5, 7, 9, 10\}$. Each of these contributes 4 to S.

   n_3 if it were nonzero, every element would contribute 9 to S.

   n_4 if it were nonzero, every element would contribute 16 to S.

   And we note that $n_1 + 4n_2 = 8 + 6 \times 4 = 32$.

   We generalize this and prove it later.

Lemma 5.4 *Let $5 \le n \le 106$. Assume $A_1, \ldots, A_m$ is an n-awesome collection of blocks where every element $y \in [106]$ appears ≤ 5 times. Let $n_1, \ldots, n_5$ be as in Notation 5.2. Then:*

1. $\sum_{i\in[5]} n_i = n$.
2. $\sum_{i\in[5]} in_i = 5m$.
3. $\sum_{i\in[5]} i^2 n_i = m^2 + 4m$.

Proof: For $1 \le i \le m$ let N_i be the set of elements that appear in i blocks. Note that $|N_i| = n_i$. It is possible that some N_i are empty.

1) The N_i form a partition of $[n]$. Hence $\sum_{i\in[5]} n_i = n$.

2) The number of incidences is $5m$. Each elements of N_i contributes i to the number of incidences. Hence $\sum_{i\in[5]} in_i = 5m$.

3) Let $S = \sum_{i\in[n]} r_i^2$. We determine $|S|$.
   Each elements of N_i contributes i^2 to S. Hence $\sum_{i\in[5]} i^2 n_i = \sum_{i\in[n]} r_i^2$.
   We now need to show that $\sum_{i\in[n]} r_i^2 = m^2 + 4m$.

We take a detour here.

There are $\binom{m}{2}$ pairs of blocks. We want to count that number a different way. For each $1 \le i \le n$ we can look at the pairs of blocks that i is in. Note that if i is in (say) A_3 and A_{12} then *nothing else is in both A_3 and A_{12}.* The number of pairs of blocks that i is in is $\binom{r_i}{2}$. (This also works if $r_i \in \{0,1\}$ since $\binom{0}{2} = \binom{1}{2} = 0$.)

$$\sum_{i\in[n]} \binom{r_i}{2} = \binom{m}{2}$$

We now do lots of algebra.

$$\sum_{i\in[n]} \binom{r_i}{2} = \frac{1}{2}\sum_{i\in[n]} r_i^2 - \frac{1}{2}\sum_{i\in[n]} r_i$$

$$\binom{m}{2} = \frac{1}{2}\sum_{i\in[n]} r_i^2 - \frac{1}{2}\sum_{i\in[n]} r_i$$

$$2\binom{m}{2} = \sum_{i\in[n]} r_i^2 - \sum_{i\in[n]} r_i$$

$$2\binom{m}{2} = \sum_{i\in[n]} r_i^2 - 5m$$

$$m^2 - m = \sum_{i\in[n]} r_i^2 - 5m$$

$$m^2 + 4m = \sum_{i\in[n]} r_i^2$$

■

The following can be derived by algebra (We used ChatGPT).

Lemma 5.5 *Let $n, n \in \mathsf{N}$. Consider the set of equations (from Lemma 5.4):*

$\sum_{i\in[5]} n_i = n$.

$\sum_{i\in[5]} i n_i = 5m$.

$\sum_{i\in[5]} i^2 n_i = m^2 + 4m$.

$(n_1, \ldots, n_5)$ *is a solution in Z^5 iff there exists $p, q \in \mathsf{Z}$ such that:*

$$\begin{aligned}
n_1 &= \frac{1}{2}m^2 - \frac{21m}{2} + 3n - p - 3q \\
n_2 &= -m^2 + 16m - 3n + 3p + 8q \\
n_3 &= \frac{1}{2}m^2 - \frac{11m}{2} + n - 3p - 6q \\
n_4 &= p \\
n_5 &= q
\end{aligned}$$

FIRST PROGRAM TO WRITE

Given n, m we wonder if there is an n-awesome collection of m blocks where every element appears ≤ 5 times.

If there is then there must exist $(n_1, n_2, n_3, n_4, n_5)$ satisfying the equations above such that there are n_i elements of $\{1, \ldots, n\}$ appear in i blocks.

This program will, given n, m, generate all possible $(n_1, n_2, n_3, n_4, n_5)$.

1. Input n, m.
2. For $p = 0$ to n

   For $q = 0$ to $n - p$

$$n_1 = \frac{1}{2}m^2 - \frac{21m}{2} + 3n - p - 3q$$
$$n_2 = -m^2 + 16m - 3n + 3p + 8q$$
$$n_3 = \frac{1}{2}m^2 - \frac{11m}{2} + n - 3p - 6q$$
$$n_4 = p$$
$$n_5 = q$$

If $(\forall i)[n_i \ge 0]$ then print $(n_1, n_2, n_3, n_4, n_5)$

END OF FIRST PROGRAM TO WRITE

The next theorem gives a lower bound on m.

Theorem 5.6 *Let $n \ge 5$.*

0) If $n \equiv 0 \pmod 4$ and $f(n) \ne *$ then $f(n) \ge \frac{n-4}{4}$.

1) If $n \equiv 1 \pmod 4$ then $f(n) \ge \frac{n-4}{4}$.

2) If $n \equiv 2 \pmod 4$ and $f(n) \ne *$ then $f(n) \ge \frac{n-2}{4}$.

3) If $n \equiv 3 \pmod 4$ and $f(n) \ne *$ then $f(n) \ge \frac{n-3}{4}$.

Proof:

Assume $A_1, \ldots, A_m$ is an n-awesome collection of blocks. For $1 \le i \le m$ let
$d_i = |A_i - \cup_{j=1}^{i-1} A_j|$
The following are obvious.

- $\sum_{i=1} d_i = n$.
- $|d_1| = 5$.
- For $2 \le i \le n$, $d_i \le 4$.
- Hence $\sum_{i=1}^{m} d_i \le 5 + 4(m-1) = 4m + 4$.

$n = \sum_{i=1} d_i \le 4m + 4$
$m \ge \frac{n-4}{4}$
Since $m \in \mathsf{N}$, $m \ge \lceil \frac{n-4}{4} \rceil$.
0) If $n \equiv 0 \pmod 4$ then $\lceil \frac{n-4}{4} \rceil = \frac{n-4}{4}$.
1) If $n \equiv 1 \pmod 4$ then $f(n) \ge \frac{n-1}{4}$ by Theorem 4.1. (Hence nothing we did in this proof is relevant.)
2) If $n \equiv 2 \pmod 4$ then $\lceil \frac{n-4}{4} \rceil = \frac{n-2}{4}$.
3) If $n \equiv 3 \pmod 4$ then $\lceil \frac{n-4}{4} \rceil = \frac{n-3}{4}$. ∎

SECOND PROGRAM TO WRITE

Given n we are trying to find $f(n)$ such that there is an n-awesome collection of $f(n)$ blocks. We want to know upper and lower bounds on $f(n)$ to constrain our search.

Use Theorem 5.6 to write a program that will, given n, find a lower bound on $f(n)$.

END OF SECOND PROGRAM TO WRITE

6 $f(n)$ for $n \ge 34$

Lemma 6.1 *Let $A_1, \ldots, A_m$ be an n-awesome collection of blocks. If every $y \in [n]$ appears in ≤ 5 block then $n \le 33$.*

Proof:

For $1 \le i \le 5$ let n_i be the number of elements that appear in exactly i blocks. By Lemma 5.5 there exists $p, q \ge 0$ such that the following holds:

$$\begin{aligned} n_1 &= \tfrac{1}{2}m^2 - \tfrac{21m}{2} + 3n - p - 3q \\ n_2 &= -m^2 + 16m - 3n + 3p + 8q \\ n_3 &= \tfrac{1}{2}m^2 - \tfrac{11m}{2} + n - 3p - 6q \\ n_4 &= p \\ n_5 &= q \end{aligned}$$

By the definition of n_i, $n_1, n_2, n_3 \ge 0$.
From $n_1 \ge 0$ we have:

$$\tfrac{1}{2}m^2 - \frac{21m}{2} + 3n - p - 3q \ge 0$$

$$3n \ge -\frac{1}{2}m^2 + \frac{21m}{2} + p + 3q$$

$$n \ge -\frac{1}{6}m^2 + \frac{21m}{6} + \frac{p}{3} + q$$

$$n \ge \frac{21m - m^2}{6} + \frac{p}{3} + q \qquad (1)$$

From $n_2 \ge 0$ we have:

$$-m^2 + 16m - 3n + 3p + 8q \geq 0$$

$$3n \leq -m^2 + 16m + 3p + 8q$$

$$n \leq -\frac{m^2}{3} + \frac{16m}{3} + p + \frac{8q}{3}$$

$$n \leq \frac{16m - m^2}{3} + p + \frac{8}{3}q \qquad (2)$$

From $n_3 \geq 0$ we have:

$$\tfrac{1}{2}m^2 - \frac{11m}{2} + n - 3p - 6q \geq 0$$

$$n \geq -\tfrac{1}{2}m^2 + \frac{11m}{2} + 3p + 6q$$

$$n \geq \frac{11m - m^2}{2} + 3p + 6q \qquad (3)$$

From (2) and (3) we have:

$$\frac{11m - m^2}{2} + 3p + 6q \leq n \leq \frac{-m^2 + 16m}{3} + p + \frac{8q}{3}$$

This implies

$$-3m^2 + 33m + 18p + 36q \leq -2m^2 + 32m + 6p + 16q$$

$$33m + 12p + 20q \leq m^2 + 32m$$

$$m + 12p + 20q \leq m^2$$

$$12p + 20q \leq m^2 - m \qquad (4)$$

We will use Equation (4) as a constrain on p, q given m in our bounding n.

Recall Inequality (2):

$$n \leq \frac{16m - m^2}{3} + p + \frac{8q}{3}.$$

We want a bound on n. Note that as m goes up, $\frac{16m-m^2}{3}$ goes down, but the constrain on p, q from Inequality (4) allows for p, q to be larger.

Here is our current problem:

Maximize $n \leq \frac{16m-m^2}{3} + p + \frac{8q}{3}$

Relative to the Constraint $12p + 20q \leq m^2 - m$ and $m, p, q \geq 0$.

Since q has a larger coefficient than p, $p + \frac{8q}{3}$ is maximized if $p = 0$. The constraint yields that

$$q \le \frac{m^2 - m}{20}.$$

Hence we have

$$n \le \frac{-m^2 + 16m}{3} + 0 + \frac{2m(m-1)}{15}$$

$$= \frac{-5m^2 + 80m}{15} + \frac{2m(m-1)}{15}$$

$$= \frac{-3m^2 + 78m}{15} = \frac{-m^2 + 26m}{15}$$

By calculus the upper bound is maximized when $m = 13$. Hence

$$n \le \frac{26 \times 13 - 13 \times 13}{5} = \frac{169}{5} = 33.8$$

Since $n \in \mathsf{N}$, $n \le 33$. ▮

Theorem 6.2 *Let $n \ge 34$.*

1. *If $n \equiv 1 \pmod 4$ then $f(n) = \frac{n-1}{4}$.*
2. *If $n \not\equiv 1 \pmod 4$ then $f(n) = *$.*

Proof: Let $A_1, \ldots, A_m$ be an n-awesome collection of blocks. By Lemma 6.1 there exists $y \in [n]$ that is in ≥ 6 of the A_i's. By Lemma 4.2 $n \equiv 1 \pmod 4$ and $m = \frac{m-1}{4}$. Hence if m *not* $\equiv 1 \pmod 4$ then there is no n-awesome collection of blocks, so $f(n) = *$. ▮

7 $f(n)$ for $5 \le n \le 33$

The next two theorems give upper bound on m, and then a third theorem easily combines them.

Lemma 7.1 *Let $A_1, \ldots, A_m$ be an n-awesome collection of blocks. If there is no $x \in A_1 \cap \cdots \cap A_m$, then $m \le 21$.*

Proof: We can assume
$A_1 = \{1, 2, 3, 4, n\}$ and
$A_2 = \{5, 6, 7, 8, n\}$.
Since some block does not have n we can assume
$A_3 = \{1, 5, 9, 10, 11\}$.
Let
$T_1 = \{i \ge 4 \colon n \in A_i\}$.
$T_2 = \{i \ge 3 \colon n \notin A_i\}$.
We prove upper bounds on $|T_1|$ and $|T_2|$.

Bound on $|T_1|$.

Let $i \in T_1$. What are the options for $A_3 \cap A_i$?

$1 \notin A_3 \cap A_i$ since then $A_1 \cap A_3 = \{1, n\}$.

$5 \notin A_3 \cap A_i$ since then $A_2 \cap A_3 = \{5, n\}$.

Hence $A_3 \cap A_i \in \{9, 10, 11\}$, so there are 3 possibilities for $A_3 \cap A_i$.

Map $i \in T_1$ to $A_3 \cap A_i$.

It is easy to see that the map is 1-1. Hence $|T_1| \leq 3$.

Bound on $|T_2|$.

Let $i \in T_2$. What are the options for $(A_1 \cap A_i, A_2 \cap A_i)$?

Since $n \notin A_i$, $A_1 \cap A_i \in \{1, 2, 3, 4\}$ and $A_2 \cap A_i \in \{5, 6, 7, 8\}$. Hence there are 16 possibilities for $(A_1 \cap A_i, A_2 \cap A_i)$.

Map $i \in T_2$ to $(A_1 \cap A_i, A_2 \cap A_i)$.

It is easy to see that this map is 1-1. Hence $|T_2| \leq 16$.

Conclusion

The number of blocks is $2 + |T_1| + |T_2| = 2 + 3 + 16 = 21$. ∎

Note 7.2 The proof of Theorem **??** has inside it an alternative proof of Lemma 4.3.

Theorem 7.3 *Let $n \in \mathsf{N}$ and $A_1, \ldots, A_m$ be an n-awesome collection. Assume that, for all $y \in [n]$, y appears in ≤ 5 of the A_i's. Then* $m \leq \left\lfloor \frac{1+\sqrt{80n+1}}{2} \right\rfloor$.

Proof:

Recall from Lemma 5.4 that we have:

$n_1 + n_2 + n_3 + n_4 + n_5 = n \quad (1)$

$n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 = 5m \quad (2)$

$n_1 + 4n_2 + 9n_3 + 16n_4 + 25n_5 = m^2 + 4m \quad (3)$.

Subtract equation (2) from equation (3):

$$2n_2 + 6n_3 + 12n_4 + 20n_5 = m^2 - m$$

$$n_2 + 3n_3 + 6n_4 + 10n_5 = \frac{m^2 - m}{2}$$

$$\frac{m^2 - m}{2} = n_2 + 3n_3 + 6n_4 + 10n_5 \leq 10(n_2 + n_3 + n_4 + n_5) \leq 10n$$

$$m^2 - m \leq 20n$$

$$m \leq \frac{1 + \sqrt{80n + 1}}{2}.$$

Since $m \in \mathsf{N}$, the upper bound follows. ∎

Theorem 7.4 *Let $n \in \mathsf{N}$ and $A_1, \ldots, A_m$ be an n-awesome collection. Assume that, for all $y \in [n]$, y appears in ≤ 5 of the A_i's. Then m is $\leq$ the min of the following quantities*

- $\left\lfloor \frac{1+\sqrt{80n+1}}{2} \right\rfloor$.
- *21.*

THIRD PROGRAM TO WRITE

Given n we are trying to find $f(n)$ such that there is an n-awesome collection of $f(n)$ blocks. We want to know upper and lower bounds on $f(n)$ to constrain our search.

Use Theorem **??** to write a program that will, given n, find an upper bound on $f(n)$.

END OF THIRD PROGRAM TO WRITE

FOURTH PROGRAM TO WRITE

Given n we are trying to find $f(n)$ such that there is an n-awesome collection of $f(n)$ blocks. We want to know upper and lower bounds on $f(n)$ to constrain our search.

We also want to, for each possible m, find all the possible $(n_1, n_2, n_3, n_4, n_5)$.

We combine the FIRST, SECOND, and THIRD programs to obtain this. Here is a sketch

1. Input(n)
2. For the upper and lower bounds on $f(n)$ using the SECOND and THIRD programs. Call them U and L.
3. For $m = L$ to U
   (a) Print(m)
   (b) Run FIRST PROGRAM on (n, m).

8 Table of Values for $f(n)$

n	$f(n)$
5	1
6	$*$
7	$*$
8	$*$
9	2
10	$*$
11	$*$
12	3
13	3
14	4
15	6
16	6
17	7
18	9
19	12
20	16
21	21
22	20
23	19
24	20
25	18
26	19
27	18
28	18
29	17
30	17
31	15
32	16
33	14

For $n \geq 34$

1. If $n \equiv 1 \pmod 4$ then $f(n) = \frac{n-1}{4}$.
2. If $n \not\equiv 1 \pmod 4$ then $f(n) = *$.

9 The Actual n-Awesome Collections Of Maximum Size

For $n = 5, 9, 12, 13$ we leave the task of finding n-awesome collections of size respectively, 1,2,3,3 to the reader. They are easy.

For $n = 14$ the max m is 4. The parameters $(n1, n2, n3, n4, n5)$ are $(8, 6, 0, 0, 0)$

$A_1 = \{1, 2, 3, 4, 5\}$
$A_2 = \{1, 6, 7, 8, 9\}$
$A_3 = \{2, 6, 10, 11, 12\}$
$A_4 = \{3, 7, 10, 13, 14\}$

For $n = 15$ the max m is 6. The parameters $(n1, n2, n3, n4, n5)$ are $(0, 15, 0, 0, 0)$.

$A_1 = \{1, 2, 3, 4, 5\}$
$A_2 = \{1, 6, 7, 8, 9\}$
$A_3 = \{2, 6, 10, 11, 12\}$
$A_4 = \{3, 7, 10, 13, 14\}$
$A_5 = \{4, 8, 11, 13, 15\}$
$A_6 = \{5, 9, 12, 14, 15\}$

For $n = 16$ the max m is 6. The parameters $(n1, n2, n3, n4, n5)$ are $(3, 12, 1, 0, 0)$.

$A_1 = \{1, 2, 3, 4, 5\}$
$A_2 = \{1, 6, 7, 8, 9\}$
$A_3 = \{1, 10, 11, 12, 13\}$
$A_4 = \{2, 6, 10, 14, 15\}$
$A_5 = \{3, 7, 11, 14, 16\}$
$A_6 = \{4, 8, 12, 15, 16\}$

For $n = 17$ the max m is 7. The parameters $(n1, n2, n3, n4, n5)$ are $(2, 12, 3, 0, 0)$.

$A_1 = \{1, 2, 3, 4, 5\}$
$A_2 = \{1, 6, 7, 8, 9\}$
$A_3 = \{1, 10, 11, 12, 13\}$
$A_4 = \{2, 6, 10, 14, 15\}$
$A_5 = \{2, 7, 11, 16, 17\}$
$A_6 = \{3, 8, 12, 14, 16\}$
$A_7 = \{3, 9, 13, 15, 17\}$

For $n = 18$ the max m is 9. The parameters $(n1, n2, n3, n4, n5)$ are $(0, 9, 9, 0, 0)$.

$A_1 = \{1, 2, 3, 4, 5\}$
$A_2 = \{1, 6, 7, 8, 9\}$
$A_3 = \{1, 10, 11, 12, 13\}$
$A_4 = \{2, 6, 10, 14, 15\}$
$A_5 = \{2, 7, 11, 16, 17\}$
$A_6 = \{3, 6, 12, 16, 18\}$
$A_7 = \{3, 8, 13, 14, 17\}$
$A_8 = \{4, 7, 13, 15, 18\}$
$A_9 = \{5, 9, 10, 17, 18\}$

For $n = 19$ the max m is 12. The parameters $(n1, n2, n3, n4, n5)$ are $(0, 0, 16, 3, 0)$.

$A_1 1 = \{1, 2, 3, 4, 5\}$
$A_2 = \{1, 6, 7, 8, 9\}$
$A_3 = \{1, 10, 11, 12, 13\}$
$A_4 = \{1, 14, 15, 16, 17\}$
$A_5 = \{2, 6, 10, 14, 18\}$
$A_6 = \{2, 7, 11, 15, 19\}$
$A_7 = \{3, 8, 13, 14, 19\}$
$A_8 = \{3, 9, 12, 15, 18\}$
$A_9 = \{4, 6, 12, 17, 19\}$
$A_{10} = \{4, 7, 13, 16, 18\}$
$A_{11} = \{5, 8, 11, 17, 18\}$
$A_{12} = \{5, 9, 10, 16, 19\}$

For $n = 20$ the max m is 16. The parameters $(n1, n2, n3, n4, n5)$ are $(0, 0, 0, 20, 0)$

$A_1 = \{1, 2, 3, 4, 5\}$
$A_2 = \{1, 6, 7, 8, 9\}$
$A_3 = \{1, 10, 11, 12, 13\}$
$A_4 = \{1, 14, 15, 16, 17\}$
$A_5 = \{2, 6, 10, 14, 18\}$
$A_6 = \{2, 7, 11, 15, 19\}$
$A_7 = \{2, 8, 12, 16, 20\}$
$A_8 = \{3, 7, 10, 17, 20\}$
$A_9 = \{3, 8, 13, 14, 19\}$
$A_{10} = \{3, 9, 12, 15, 18\}$
$A_{11} = \{4, 6, 12, 17, 19\}$
$A_{12} = \{4, 7, 13, 16, 18\}$
$A_{13} = \{4, 9, 11, 14, 20\}$
$A_{14} = \{5, 6, 13, 15, 20\}$
$A_{15} = \{5, 8, 11, 17, 18\}$
$A_{16} = \{5, 9, 10, 16, 19\}$

For $n = 21$ the max m is 21. The parameters $(n1, n2, n3, n4, n5)$ are $(0, 0, 0, 0, 21)$.

$$
\begin{aligned}
A_1 &= \{1, 2, 7, 9, 19\},\\
A_2 &= \{2, 3, 8, 10, 20\},\\
A_3 &= \{3, 4, 9, 11, 21\},\\
A_4 &= \{1, 4, 5, 10, 12\},\\
A_5 &= \{2, 5, 6, 11, 13\},\\
A_6 &= \{3, 6, 7, 12, 14\},\\
A_7 &= \{4, 7, 8, 13, 15\},\\
A_8 &= \{5, 8, 9, 14, 16\},\\
A_9 &= \{6, 9, 10, 15, 17\},\\
A_{10} &= \{7, 10, 11, 16, 18\},\\
A_{11} &= \{8, 11, 12, 17, 19\},\\
A_{12} &= \{9, 12, 13, 18, 20\},\\
A_{13} &= \{10, 13, 14, 19, 21\},\\
A_{14} &= \{1, 11, 14, 15, 20\},\\
A_{15} &= \{2, 12, 15, 16, 21\},\\
A_{16} &= \{1, 3, 13, 16, 17\},\\
A_{17} &= \{2, 4, 14, 17, 18\},\\
A_{18} &= \{3, 5, 15, 18, 19\},\\
A_{19} &= \{4, 6, 16, 19, 20\},\\
A_{20} &= \{5, 7, 17, 20, 21\},\\
A_{21} &= \{1, 6, 8, 18, 21\}.
\end{aligned}
$$

For $n = 22$ the max m is 20. The parameters $(n1, n2, n3, n4, n5)$ are $(1, 1, 1, 1, 18)$.

For $n = 23$ the max m is 19. The parameters $(n1, n2, n3, n4, n5)$ are $(2, 2, 3, 0, 16)$.

For $n = 24$ the max m is 20. The parameters $(n1, n2, n3, n4, n5)$ are $(5, 0, 0, 0, 19)$.

For $n = 25$ the max m is 18. The parameters $(n1, n2, n3, n4, n5)$ are $(6, 1, 4, 0, 14)$.

For $n = 26$ the max m is 19. The parameters $(n1, n2, n3, n4, n5)$ are $(8, 1, 0, 0, 17)$.

For $n = 27$ the max m is 18. The parameters $(n1, n2, n3, n4, n5)$ are $(9, 3, 0, 0, 15)$.

For $n = 28$ the max m is 18. The parameters $(n1, n2, n3, n4, n5)$ are $(12, 0, 1, 0, 15)$.

For $n = 29$ the max m is 17. The parameters $(n1, n2, n3, n4, n5)$ are $(14, 0, 2, 0, 13)$.

For $n = 30$ the max m is 17. The parameters $(n1, n2, n3, n4, n5)$ are $(16, 0, 0, 1, 13)$.

For $n = 31$ the max m is 15. The parameters $(n1, n2, n3, n4, n5)$ are $(18, 2, 1, 0, 10)$.

For $n = 32$ the max m is 16. The parameters $(n1, n2, n3, n4, n5)$ are $(20, 0, 0, 0, 12)$.

For $n = 33$ the max m is 14. The parameters $(n1, n2, n3, n4, n5)$ are $(23, 1, 0, 0, 9)$.