Cheat Sheet For $\mathsf{R}^n \to (\ell_2, \ell_{2^{dn}})$, $\mathsf{R}^n \not\to (\ell_2, \ell_{d'n})$

1 Goal

1. (Szlam [5]) $\exists d$, $\mathsf{R}^n \to (\ell_2, \ell_{2^{dn}})$.
2. (Conlon & Fox [1]), $\exists d'$ $\mathsf{R}^n \not\to (\ell_2, \ell_{2^{d'n}})$.

2 Frankl-Wilson Set Systems: OddTown

This is an Easy case of the next theorem.

Theorem 2.1 *(Oddtown Theorem) Let $n \in \mathsf{N}$ such that $k \le n$. Let $F_1, \ldots, F_L \in 2^{[n]}$ be such that:*

$$(\forall 1 \le i \le L)[|F_i| \text{ odd }].$$

$$(\forall 1 \le i < j \le s)[|F_i \cap F_j| \text{ even }].$$

Then $L \le n$.

Proof: For $1 \le i \le L$ let f_i be the bit vector for F_i. Note that f_i is a vector of n bits, k of which are 1's. Let

$$f_i = (f_{i1}, \ldots, f_{in}).$$

Note that $f_{ij} = 1$ iff $j \in F_i$.

We view the f_i's as n-dimensional vectors over $F_2 = \{0,1\}$ so the arithmetic is mod 2.

We show that the f_i's are linearly independent, hence there are at most n of them, so $L \le n$.

Claim The f_i's are linearly independent $\pmod 2$.

Proof:

$$\begin{aligned} f_i \cdot f_j = &\ f_{i1}f_{j1} + f_{i2}f_{j2} + \cdots f_{in}f_{jn} \\ = &\ |F_i \cap F_j|. \end{aligned}$$

Since $|F_i \cap F_j|$ is even, and $|F_i|$ is odd, we have

$$f_i \cdot f_j \pmod 2 = \begin{cases} 0 \text{ if } i \ne j; \\ 1 \text{ if } i = j. \end{cases} \tag{1}$$

Let $\lambda_1, \ldots, \lambda_L$ be such that

$$\lambda_1 f_1 + \cdots + \lambda_L f_L = 0.$$

Let $1 \le i \le L$. Dot both sides by f_i to get $\lambda_i = 0$

Hence, for every $1 \le i \le L$, $\lambda_i = 0$.

End of Proof of Claim ∎

3 FW Set Systems: What We Need

We will use, but not proof, the following, which is in the FW paper. The proof is similar but harder. I plan to have it written up.

Theorem 3.1 *Let $k, n \in \mathsf{N}$. Let q be a prime power.*

1. *Let $F_1, \ldots, F_L \in \binom{[n]}{k}$ be such that:*
$$(\forall 1 \le i < j \le s)[|F_i \cap F_j| \not\equiv k \pmod q].$$
Then $L \le \binom{n}{q-1}$.

2. *If $F_1, \ldots, F_{\binom{[n]}{q-1}+1} \in \binom{[n]}{k}$ then there exists $1 \le i < j \le \binom{[n]}{q-1} + 1$ with $|F_i \cap F_j| \equiv k \pmod q$. (This is the contrapositive of Part 1.)*

3. *If $F_1, \ldots, F_{\binom{[n]}{q-1}+1} \in \binom{n}{2q-1}$ then there exists $1 \le i < j \le \binom{[n]}{q-1} + 1$ with $|F_i \cap F_j| = q - 1$. (This is Part 2 with $k = 2q-1$ coupled with the observation that if $|F_i \cap F_j| \equiv 2q-1 \pmod q$ then $|F_i \cap F_j| = q-1$ since otherwise $F_i = F_j$.)*

Example 3.2 $n = 10$ and $q = 3$. Then Part 3 says.
If $F_1, \ldots, F_{\binom{[10]}{2}+1} \in \binom{[10]}{5}$ then there exists $1 \le i < j \le \binom{10}{2}$ with $|F_i \cap F_j| = 2$.
Lets use concrete numbers:
If $F_1, \ldots, F_{46} \in \binom{[10]}{5}$ then there exists $1 \le i < j \le \binom{10}{2} = 45$ with $|F_i \cap F_j| = 2$.
Lets use concrete numbers:

4 A Bound on the Chromatic Number of R^{10}

We will derive a value of c, which we make as large as possible, so that for all $\mathrm{COL}\colon \mathsf{R}^{10} \to [c]$ there exists $u, v \in \mathsf{R}^{10}$ with $d(u, v) = 1$.

We restrict COL to the following set S of vectors such that:

- 5 of the coordinates are 0
- 5 of the coordinates are $\frac{1}{\sqrt{6}}$.
- Note that $|S| = \binom{10}{5} = 252$.

Let T map S to $\binom{[10]}{5}$ by using bit vectors with $\frac{1}{\sqrt{6}}$ instead of 1.

For example $T(\frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, 0, 0, 0, \frac{1}{\sqrt{6}},) = \{1, 3, 4, 6, 10\}$.

Let $u, v \in S$ such that they have $|S(u) \cap S(v)| = 2$. Then

- There are 2 coordinates where u and v both have $\sqrt{\frac{1}{sqrt6}}$.
- There are 3 coordinates where u is $\frac{1}{\sqrt{6}}$ and v is 0.
- There are 3 coordinates where v is $\frac{1}{\sqrt{6}}$ and u is 0.
- There are 2 coordinates where v is 0 and u is 0.

Hence

$$d(u, v) = \sqrt{(\frac{1}{\sqrt{6}})^2 + (\frac{1}{\sqrt{6}})^2 + (\frac{1}{\sqrt{6}})^2 + (\frac{1}{\sqrt{6}})^2 + (\frac{1}{\sqrt{6}})^2 + (\frac{1}{\sqrt{6}})^2} = 1.$$

Hence we need to show that two of the elements in the image of T intersect in two positions. Note that *this is now a problem about set systems!*

Let the image of T be $F_1, \ldots, F_{252}$. By Example 3.2 within any set of 46 of these there are two that intersect in 2 places. Now we look at the number of colors. If there are c colors then there will be some set of $\lceil \frac{252}{c} \rceil$ that are the same color. So we need the max c such that $\lceil 252 \rceil c \geq 46$. We take $c = 5$.

5 The Chromatic Number of R^n

Definition 5.1 For $n \geq 2$, $c(n)$ is the chromatic number of R^n

Theorem 5.2

1. *$\forall n$, $c(n) > \max_{q \text{ prime power}} \frac{\binom{n}{2q-1}}{\binom{n}{q-1}+1}$.*

2. *$\exists d\ \forall n$, $c(n) \geq 2^{dn}$. Follows from Part 1.*

Proof:
1) Let $S \subseteq \mathsf{R}^n$ be all of the vectors such that

- $n - 2q - 1$ of the components are 0.
- $2q - 1$ of the components are $\frac{1}{\sqrt{2q}}$.

Let $F \colon S \to \binom{[n]}{2q-1}$ by viewing each vector in S as a bit vector though with $\frac{1}{\sqrt{2q}}$ instead of 1.
Claim Let $u, v \in S$. If $|F(u) \cap F(v)| = f$ then $d(u, v) = 2 - \frac{f+1}{q}$. Hence $d(u, v) = 1$ iff $|F(u) \cap F(v)| = q - 1$.
Proof of Claim: Assume $|F(u) \cap F(v)| = f$ then:

- There are f coordinates where u and v both have $\frac{1}{\sqrt{2q}}$.
- There are $2q - 1 - f$ coordinates where u has $\frac{1}{2q}$ and v has 0.
- There are $2q - 1 - f$ coordinates where v has $\frac{1}{2q}$ and u has 0.
- There are $n - 4q + f + 2$ coordinates where u and v are both 0.

Hence $d(u, v) = 2 \times (2q - 1 - f) \times \frac{1}{2q} = \frac{2q-1-f}{q} = 2 - \frac{f+1}{q}$.
End of Proof of Claim
Restrict COL to S. Since $|S| = \binom{n}{2q-1}$ and there are c colors, some color must occur $\geq \binom{n}{2q-1}/c = \binom{n}{q-1} + 1$ times. Let S' be the subset of S that has that color. Since $S' \subseteq \binom{[n]}{2q-1}$ and $|S'| \geq \binom{n}{q-1} + 1$, by Theorem 3.1.3, there exists two elements of S with intersection of size $q - 1$. Let those two elements be $F(u)$ and $F(v)$. Since $|F(u) \cap F(v)| = q - 1$, by the Claim, $d(u, v) = 1$.
$\blacksquare$

6 There exists d Such That $\mathsf{R}^n \rightarrow (\ell_2, \ell_{2^{dn}}))$

Theorem 6.1 *(Szlam [5]) There exists d such that $\mathsf{R}^n \rightarrow (\ell_2, \ell_{2^{dn}})$.*

Proof: By Theorem 5.2 there exists d such that $c(n) > 2^{dn}$. Thats the d that we use. Let $m = 2^{dn}$.

We will need the following notation: $\vec{1}$ is the vector $(1, 0, \ldots, 0)$ in R^n.

Let COL: $\mathsf{R}^n \rightarrow [2]$.

Case 1 There is a BLUE ℓ_m. Done

Case 2 There is no BLUE ℓ_m. We form a coloring COL: $\mathsf{R}^n \rightarrow [m]$ as follows:

Given point $p \in \mathsf{R}^n$ look at

$$p + \vec{1}, p + 2\vec{1}, \ldots, p + m\vec{1}.$$

Since there is no BLUE ℓ_m, there exists i such that $\mathrm{COL}(p + i\vec{1})$ is RED. Color p with the least such i.

By Theorem 5.2 there exists points $u, v \in \mathsf{R}^n$ and $1 \leq i \leq m$ such that $d(u, v) = 1$ and u, v are the same color. Hence $u + i\vec{1}$ and $v + i\vec{1}$ are both RED. Since $d(u, v) = 1$, $d(u + i\vec{1}, v + i\vec{1}) = 1$. Hence $u + i\vec{1}$ and $v + i\vec{1}$ form a RED ℓ_2. ∎

7 Lemmas Needed for $\exists d$, $\mathsf{R}^n \not\to (\ell_2, \ell_{2^{dn}})$:$t$-Separated

We will be 2-coloring the $m \times m$ square and then use that to form a periodic coloring of R^2. Hence we think of coloring the $m \times m$ square with the two horizontal sides identified and the new vertical sides identified. We denote this T_m^2. (The T is for Taurus.) (TO BILL: this should be torus)

We need several lemmas.

Definition 7.1 Let $t \in \mathsf{R}^+$. Let $P \subseteq T_m^2$.

1. P is t-separated if, for all $p, q \in P$, $d(p,q) \geq t$.
2. P is *maximally t-separated* (1) if P is t-separated and (2) for all $r \notin P$, $P \cup \{r\}$ is not t-separated.

Lemma 7.2 *Let $t \in \mathsf{R}^+$ and $m \in \mathsf{N}$.*

1. *There exists $P \subseteq T_m^2$ that is maximally t-seperated.*
2. *If $P \subseteq T_m^2$ is maximally t-seperated then $|P| \leq \frac{(m/t)^2}{\pi}$.*
3. *If $P \subseteq T_m^2$ is maximally $\frac{1}{3}$-seperated then $|P| \leq (1.7m)^2$. This follows from Part 2.*

Proof:
1) A greedy algorithm forms a maximally t-seperated set.
2) Let $p \in P$. Then there is no element of P inside the circle centered at p of radius t. This circle has area πt^2. The set T_m^2 has area m^2. Hence

$|P| \times \pi t^2 \leq m^2$, so $|P| \leq \frac{(m/t)^2}{\pi}$. ∎

Lemma 7.3 *Let $t \in \mathsf{R}^+$. Let $S \subseteq \mathsf{R}^2$ be t-seperated. Let $\vec{p} \in \mathsf{R}^2$. Let $s \geq 0$. The number of points of S within s of $\vec{p}$ is at most $(2s/t+1)^2$.*

Proof: Let T be the set of points within t of $\vec{p}$. For every $\vec{q} \in T$ we look at the circle centered at $\vec{q}$ of radius $t/2$ (we can't use radius t since then the circles would not be disjoint). These circles have no other points of T in them and are disjoint. These circles have area $\pi(t/2)^2$. The union of these circles is contained in the circle around $\vec{p}$ of radius $s+t/2$ which has area $\pi(s+t/2)^2$. Hence

$|T| \times \pi t^2/4 \leq \pi(s+t/2)^2$

$|T| \times (t/2)^2 \leq (s+t/2)^2$

$|T| \leq (\frac{s+t/2}{t/2})^2 = (2s/t+1)^2$. ∎

8 Lemmas Needed for $\exists d$, $\mathsf{R}^n \not\to (\ell_2, \ell_{2^{dn}})$:Vornoi

Definition 8.1 Assume $S \subseteq \mathsf{R}^2$ or $S \subseteq T_2^m$. If $p \in S$ then V_p is the set of points of R^2 or T_2^m that are closer (or tied) to p then to any other point of S. The *Voronoi Diagram of* S is the set of all the V_p's.

Note 8.2 There exists $S \subseteq \mathsf{R}^n$ and an $s \in S$ such that V_p is a convex $|S|$-gon.

Lemma 8.3 *Let $S \subseteq \mathsf{R}^2$ be a maximal t-separated set. We form the Voronoi diagram of S. The Voronoi cells are $\{V_p\}_{p \in S}$.*

1. *If $x \in V_p$ then $d(x,p) \le t$.*
2. *If $p, p' \in V_p$ then $d(p,p') \le 2t$. (This follows from Part 1.)*
3. *If $p, p' \in S$ and V_p, $V_{p;}$ share a boundary then $d(p,p') \le 2t$.*
4. *V_p is convex polygon with ≤ 25 sides.*

Proof:
1) Assume, by way of contradiction, that there is an $x \in V_p$ and $d(x,p) > t$. Since $x \in V_p$, $d(x,p)$ is the smallest distance from x to a point of S. Hence x is greater than t away from any point in S. Since S is maximal, $x \in S$ which is a contradiction.
3) Draw a line from p to p'. It will hit a point x that is on both the boundary of V_p and the boundary of $V_{p'}$. By Part 1
$d(p,p') = d(p,x) + d(x,p') \le t + t = 2t$.
4) V_p is a convex polygon. Map each side of V_p to the p' such that V_p and $V_{p'}$ share that side. Using Part 2 we get that the number of sides is bounded above by the number of points of $p' \in S$ such that $d(p,p') \le 2t$. By Lemma 7.3 the number of such points is $\le ((2 \times 2t)/t + 1)^2 = 5^2 = 25$. ∎

Lemma 8.4 *Let K be a 1-seperated set. Let $s \ge 1$. There is a set $K' \subseteq K$ that is s-separated such that $|K'| \ge |K|/(2s+1)^2$.*

9 There exists d', $\mathsf{R}^n \not\to (\ell_2, \ell_{2^{d'n}})$

Theorem 9.1 *There exists d' such that $\mathsf{R}^2 \not\to (\ell_2, \ell_{2^{d'n}})$.*

Proof: Let P be a maximal $\frac{1}{3}$-separated subset of T_2^m. We create the Voronoi diagram of P.

Let $Q \subseteq P$ be formed by, for each $p \in P$, choose it with probability x (we will determine x later).

Let $S \subseteq Q$ be the set of points $s \in Q$ such that, for all $s' \in Q$, $d(s, s') > 5/3$.

Recall that we have a Voronoi diagram formed by the points in P. Let the Voronoi cells that have a point of S in them be denoted $V_1, \ldots, V_{|S|}$.

We will color each V_i, including boundary, RED. We will color every other point in T_2^m BLUE. We will then use this to periodically color R^2. We view this as tiling the plane with $m \times m$ tiles and coloring all the tiles the same.

We will show that if you take a nine tiles arrange 3×3 then there is no RED ℓ_2 or BLUE ℓ_m with a point in the middle tile. This will suffice.

No RED ℓ_2 This part does not use probability.

Let q, q' both be RED.

Case 1: q, q' are in the same Voronoi cell. By Lemma 8.3.2 $d(q, q') \le 1/3$.

Case 2: q, q' are in the same tile but in different Voronoi cells. Let the Voronoi cells have centers p, p'. Then

$$d(p, p') \le d(p, q) + d(q, q') + d(q', p') \le \frac{1}{3} + 1 + \frac{1}{3} = \frac{5}{3}.$$

But by definition of S, $d(p, p') > \frac{5}{3}$.

Case 3: q, q' are in different tiles but in the analogous Voronoi cells. Let the Voronoi cells have centers p, p'. Since $d(p, p') = m$, $d(q, q') \ge m - \frac{1}{3} > 1$.

Case 4: q, q' are in different tiles and non-analogous Voronoi cells. Since the Voronoi diagram was on a Taurus this is identical to Case 2.

No BLUE ℓ_m

Let $L = (q_1, \ldots, q_m)$ be an ℓ_m. We bound the probability that L is BLUE.

Let $\{p_i\}_{i=0}^{m'-1}$ be such that, for $0 \le i \le m'-1$, $q_i \in V_{p_i}$. We need to bound the probability that V_{p_i} is BLUE. Not so fast! We need to show that all of the V_{p_i} are distinct.

Let $q, q' \in \{q_0, \ldots, q_{m'-1}\}$. Let $\{p, p'\}$ be such that $q \in V_p$ and $q' \in V_{p'}$.

Case 1 q, q' are in the same tile and in the same Voronoi cell. This cannot happen since $d(q, q') \ge 1$ and by Lemma 8.3.2 the diameter of these cells is $2/3$.

Case 2 q, q' are in the different tiles but in analogous Voronoi cells. Two points in analogous cells are at least $m - \frac{2}{3}$ apart. Since $d(q, q') \le m - 1$, q, q' cannot be in different tiles but in analogous Voronoi cells.

The probability that L is BLUE is the prob that V_{p_1}, V_{p_2}, …, V_{p_m} are all BLUE.

Let $p \in P$. We determine a lower bound on the probability that V_p is RED. Recall that V_p is RED iff $p \in S$.

BILL TO BILL- I NEED TO FINISH THIS. IT REQUIRES THAT LEMMA ABOUT SIGN PATTERNS. ∎

References

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