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Extended VDWs Theorem

Exposition by William Gasarch

January 23, 2025

VDW and Extended VDW

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$\mathrm{COL}: [W(X + 1, c)] \to [c]$. By VDW there exists A, D
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$A, A+D, \ldots, A+(k-1)D$ are color CCC. So $\mathrm{COL}(D) \ne CCC$.
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$A, A+\frac{XD}{k-1}, A+\frac{2XD}{k-1}, \ldots, A+\frac{(k-1)XD}{k-1}$. So $\mathrm{COL}(\frac{XD}{k-1}) \ne CCC$.

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$A, A+\frac{XD}{k-1}, A+\frac{2XD}{k-1}, \ldots, A+\frac{(k-1)XD}{k-1}$. So $\mathrm{COL}(\frac{XD}{k-1}) \ne CCC$.
$D, 2D, \ldots, \frac{X}{k-1}D$ not colored CCC, only use $c-1$ colors.

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This is an exercise.