1 Eliminating some $(n_1, n_2, n_3.n_4, n_5)$

We give several examples of times when $n, m, n_1, \ldots, n_5$ satisfy the equations yet there is no n-awesome set of m blocks where, for all $1 \le i \le 5$, there are n_i elements of $[n]$ that occur in i blocks. We will also give general theorems and programs-to-write.

1.1 Helpful Lemma

Definition 1.1 (Reminder) Let $n, m \in \mathsf{N}$. Assume that there is an n-awesome collection of m blocks where every $y \in [n]$ appears ≤ 5 times.

1. For $1 \le i \le 5$, n_i is the number of elements of $\{1, \ldots, n\}$ that are in i blocks.
2. For $y \in \{1, \ldots, n\}$, r_y is the number of blocks that y is in. We refer to r_y as *the multiplicity of* y. Note that $1 \le r_y \le 5$.

Definition 1.2 Let $n, m \in \mathsf{N}$. Assume that there is an n-awesome collection of m blocks where every $y \in [n]$ appears ≤ 5 times. Let n_i, r_y be as in Definition 1.1.

1. The collection is *of type* $(n_1, n_2, n_3, n_4, n_5)$.
2. Let A be a block with elements $\{a, b, c, d, e\}\}$. Assume $r_a \ge r_b \ge r_c \ge e_d \ge e_e$. Then we say the block is *of type* $[r_a, r_b, r_c, r_d, r_e]$. We will be using $[b_1, b_2, b_3, b_4, b_5]$ for our notation for block types.

Lemma 1.3 *Let $n, m \in \mathsf{N}$. Assume that there is an n-awesome collection of m blocks where every $y \in [n]$ appears ≤ 5 times. Let n_i and r_y be as in Definition 1.1. Let A be a block.*

1. $\sum_{y \in A} r_y = m + 4$
2. *The block type of A is a tuple $[b_1, b_2, b_3, b_4, b_5]$ where $\sum_{y \in A} b_i = m + 4$. (This follows from Part 1.)*

Proof: Let

$$\mathcal{B} = \{(B, x) \colon B \text{ is a block, } B \ne A, \text{and } x \in A \cap B\}.$$

Since every block intersects A exactly once, $|\mathcal{B}| = m - 1$.

Another way to compute $|\mathcal{B}|$ is to see how many other blocks each of the elements in A is in. Hence

$$\sum_{y \in A} (r_y - 1) = |\mathcal{B}| = m - 1$$

$$\sum_{y \in A} (r_y - 1) = m - 1$$

$$\sum_{y \in A} r_y = m + 4$$

∎

FIFTH PROGRAM TO WRITE

We will need to be find possible block types.

1. Input m
2. Output all $[b_1, b_2, b_3, b_4, b_5]$ such that the following hold:
   - $b_1 \ge b_2 \ge b_3 \ge b_4 \ge b_5$.
   - $b_1 + b_2 + b_3 + b_4 + b_5 = m + 4$.
   - For all y, $1 \le b_y \le 5$.

END OF FIFTH PROGRAM TO WRITE

1.2 Simple Examples

In all of the examples we assume there is an n-awesome collection of m blocks such that, for all $y \in [n]$, $r_y \le 5$. Oddly enough, n will not be needed in the examples.

1. $m = 20$, $(n_1, n_2, n_3, n_4, n_5)$ is such that $n_1 \ge 1$ or $n_2 \ge 1$ or $n_3 \ge 1$.

   Let A be any block. We know that $\sum_{y \in A} r_y = m + 4 = 24$ The only possible block type is $[5, 5, 5, 5, 4]$. Hence A has elements of multiplicities 4 and 5. Therefore A has no elements of multiplicty 1 or 2 or 3. This contradicts $n_1 \ge 1$ or $n_2 \ge 1$ or $n_3 \ge 1$. Hence there is no n-awesome collection of 20 blocks such that every $y \in [n]$ has $r_y \le 5$, and has one of n_1, n_2, n_3 at least 1.

2. $m = 18$, $(n_1, n_2, n_3, n_4, n_5)$, $n_1 \ge 1$ or $n_2 \ge 1$.

   Let A be any block. We know that $\sum_{y \in A} r_y = m + 4 = 22$ The only possible block types are $[5, 5, 5, 5, 2]$, $[5, 5, 5, 4, 3]$, $[5, 5, 4, 4, 4]$. Hence A has no elements of multiplicty 1 or 2. This contradicts $n_1 \ge 1$ or $n_2 \ge 1$. Hence there is no n-awesome collection of 18 blocks such that every $y \in [n]$ has $r_y \le 5$, and has one of n_1, n_2 at least 1.

3. We generalize the last two examples with a theorem whose proof we leave to the reader.

   Theorem 1.4 *Let $n, m \in \mathsf{N}$. Let $(n_1, n_2, n_3, n_4, n_5)$ be such that*

   (a) There is an $1 \le i \le 5$ such that $n_i \ge 1$.

   (b) If Z is a multisets of 5 numbers from $\{1, 2, 3, 4, 5\}$ that sum to $m + 4$ then $i \notin Z$.

   Then there is no n-awesome collection of m sets such that (a) for all $y \in [n]$, $r_y \le 5$, and (b) the collection is of type $(n_1, n_2, n_3, n_4, n_5)$.

SIXTH PROGRAM TO WRITE

We will use Theorem 1.4 go generate screen out some $(n_1, n_2, n_3, n_4, n_5)$.

(a) Input n, m

(b) Run Program FIFTH to find all possible block types $[b_1, b_2, b_3, b_4, b_5]$

(c) Run FIRST program but at the end also test if $(n_1, n_2, n_3, n_4, n_5)$ has some number that is in NONE of the block types. If so, then when you print it out also add

Not a collection type since n_i is not in any of the block types.

END OF SIXTH PROGRAM TO WRITE

4. $m = 17$, $(n_1, n_2, n_3, n_4, n_5)$ and $n_2 = 0$ and $n_4 = 1$. Let A be any block. We know that $\sum_{y \in A} r_y = m + 4 = 21$. The only block types are $[5, 5, 5, 5, 1]$, $[5, 5, 5, 3, 3]$, $[5, 5, 4, 4, 3]$, $[5, 4, 4, 4, 4]$.

   No block can use be of the last two types since $n_4 = 1$ and they both have 2 elements of multiplicity 4. Hence all blocks are of the first two types. But then A has no elements of multiplicty 4. This contradicts $n_4 = 1$. Hence there is no n-awesome collection of 17 blocks such that every $y \in [n]$ has $r_y \leq 5$, and has $n_2 = 0$ and $n_4 = 1$.

5. One could have a general theorem and a program based on the last example. However, we will not. Instead we will present a much more general technique in the next section.

2 The Equations Approach: An Example, A Theorem, and a Program

Example: $n = 22$, $m = 17$, $(n_1, n_2, n_3, n_4, n_5) = (0, 3, 7, 2, 10)$.

Let A be any block. We know $\sum_{y \in A} r_y = m + 4 = 21$

The only possible blocktypes are:

$[5, 5, 5, 4, 2]$, $[5, 5, 5, 3, 3]$, $[5, 5, 4, 4, 3]$, $[5, 4, 4, 4, 4]$.

No block can use be of the last types since $n_4 = 2$.

Let x_1 be the number of blocks of type $[5, 5, 5, 4, 2]$.

Let x_2 be the number of blocks of type $[5, 5, 5, 3, 3]$.

Let x_3 be the number of blocks of type $[5, 5, 4, 4, 3]$.

There are 3 elements of multiplicity 2. Hence they will appear $2n_2 = 6$ times.

There are 7 elements of multiplicity 3. Hence they will appear $3n_3 = 21$ times.

There are 2 elements of multiplicity 4. Hence they will appear $4n_4 = 8$ times.

There are 10 elements of multiplicity 5. Hence they will appear $5n_5 = 50$ times.

Hence

The only way to get 2's is with block of type $[5, 5, 5, 4, 2]$. Since there are 6 2's, $x_1 = 6$

The only way to get 3's is with blocks of type either $[5, 5, 5, 3, 3]$ (they provide 2 3's) and blocks of type $[5, 5, 4, 4, 3]$ (they provide 1 3).

Hence $2x_1 + x_3 = 21$

By similar reasoning with 4's: $x_1 + 2x_3 = 4$

By similar reasoning with 5's $3x_1 + 3x_2 + 2x_3 = 50$.

We also know that there are 17 blocks so $x_1 + x_2 + x_3 = 17$.

To summarize:

$x_1 = 6$

$2x_1 + x_3 = 21$

$x_1 + 2x_3 = 4$

$3x_1 + 3x_2 + 2x_3 = 50$

$x_1 + x_2 + x_3 = 17$.

We can eliminate x_1 by setting x_1 to 6:

$12 + x_3 = 21$

$6 + 2x_3 = 4$

$18 + 3x_2 + 2x_3 = 50$

$6 + x_2 + x_3 = 17$

Cleaning these up we get

$x_3 = 9$

$2x_3 = -2$

$3x_2 + 2x_3 = 32$

$x_2 + x_3 = 11$

These clearly have no solution in N.

The example inspires the following theorem whose proof we leave to the reader.

Theorem 2.1 *Let $n, m, n_1, n_2, n_3, n_4, n_5 \in \mathsf{N}$ such that $\sum_{i=1}^{n} n_i = n$. Let $T_1, \ldots, T_L$ be all 5-tuples in monotone decreasing order that sum to $m+4$ (so potential blocktypes). Consider the following 6 linear equations in L variables:*

For $1 \le i \le L$, $1 \le j \le 5$, let t_i^j be the number of j's in T_i. The equations are

$x_1 t_1^1 + x_2 t_2^1 + \cdots + x_L t_L^1 = 1 \times n_1$.

$x_1 t_1^2 + x_2 t_2^2 + \cdots + x_L t_L^2 = 2 \times n_2$.

$x_1 t_1^3 + x_2 t_2^3 + \cdots + x_L t_L^3 = 3 \times n_3$.

$x_1 t_1^4 + x_2 t_2^4 + \cdots + x_L t_L^4 = 4 \times n_4$.

$x_1 t_1^5 + x_2 t_2^5 + \cdots + x_L t_L^5 = 5 \times n_5$.

$x_1 + x_2 + \cdots + x_L = m$

If this set of equations has no solution in N then there is no n-awesome collection of m blocks where every $y \in [n]$ occurs ≤ 5 times of type $(n_1, n_2, n_3.n_4, n_5)$.

SEVENTH PROGRAM TO WRITE

We want to, given n, m, $(n_1, n_2, n_3, n_4, n_5)$ form the set of equations similar to those in the example.

Here is a sketch

1. Input $n, m, n_1, n_2, n_3, n_4, n_5$
2. Use the FIFTH program to find all block types $T_1, \ldots, T_L$.
3. We will set up 6 linear equations in L variables.

   For $1 \le i \le L$, $1 \le j \le 5$, let t_i^j be the number of j's in T_i. The equations are

   $x_1 t_1^1 + x_2 t_2^1 + \cdots + x_L t_L^1 = 1 \times n_1$.

   $x_1 t_1^2 + x_2 t_2^2 + \cdots + x_L t_L^2 = 2 \times n_2$.

   $x_1 t_1^3 + x_2 t_2^3 + \cdots + x_L t_L^3 = 3 \times n_3$.

   $x_1 t_1^4 + x_2 t_2^4 + \cdots + x_L t_L^4 = 4 \times n_4$.

   $x_1 t_1^5 + x_2 t_2^5 + \cdots + x_L t_L^5 = 5 \times n_5$.

   $x_1 + x_2 + \cdots + x_L = m$
4. Find all solutions to these equations in N (includes 0).
5. Output the equations. If they have a solution, output all solutions.. If they do not have a solution, output that there is no solution.

END OF SEVENTH PROGRAM

3 The Equations and Pairs Method: Examples, Theorem, and Program

Example 1 $m = 17$, $(n_1, n_2, n_3, n_4, n_5) = (1, 1, 7, 4, 9)$

Let A be any block. We know that $\sum_{y \in A} r_y = m + 4 = 21$

The only possible block types are $[5,5,5,5,1]$, $[5,5,5,4,2]$, $[5,5,5,3,3]$, $[5,5,4,4,3]$, $[5,4,4,4,4]$.

x_1: $[5,5,5,5,1]$ x_2: $[5,5,5,4,2]$ x_3:$[5,5,5,3,3]$.

x_4:$[5,5,4,4,3]$. x_5:$[5,4,4,4,4]$.

My similar reasoning as in the last example we have:

There are $n_1 = 1$ 1.

There are $2n_2 = 2$ 2's.

There are $3n_3 = 21$ 3's.

There are $4n_4 = 16$ 4's.

There are $5n_5 = 45$ 5's.

Hence

$x_1 = 1$

$x_2 = 2$

$2x_3 + x_4 = 21$

$x_2 + 2x_4 + 4x_5 = 16$

$4x_1 + 3x_2 + 3x_3 + 2x_4 + x_5 = 45$

$x_1 + x_2 + x_3 + x_4 + x_5 = 17$

This has four solutions

$(1, 2, 7, 7, 0)$

$(1, 2, 8, 5, 1)$

$(1, 2, 9, 3, 2)$

$(1, 2, 10, 1, 3)$

We consider each solution:

Case 1: $(1, 2, 7, 7, 0)$.

Recall that $n_4 = 4$. Hence the number of pairs of elements of multiplicty 4 is $\binom{4}{2} = 6$.

The $x_4 = 7$ block of type $[5,5,4,4,3]$ contributes $7 \times \binom{2}{2} = 7$ pairs. This is a contradiction.

Case 2: $(1, 2, 8, 5, 1)$.

The $x_4 = 5$ block of type $[5,5,4,4,3]$ contributes $5 \times \binom{2}{2} = 5$ pairs.

The $x_5 = 1$ block of type $[5,4,4,4,4]$ contributes $1 \times \binom{4}{2} = 6$ pairs. This is a contradiction.

Case 3: $(1, 2, 9, 3, 2)$.

The $x_5 = 2$ block of type $[5,4,4,4,4]$ contributes $2 \times \binom{4}{2} = 12$ pairs. This is a contradiction.

Case 4: $(1, 2, 10, 1, 3)$.

The $x_5 = 3$ block of type $[5,4,4,4,4]$ contributes $3 \times \binom{4}{2} = 18$ pairs. This is a contradiction.

Example 2 $m = 17$, $(0, 4, 4, 5, 9)$.

Let A be any block. We know that $\sum_{y \in A} r_y = m + 4 = 21$.

The only possible block types are

$[5,5,5,4,2]$-x_1 , $[5,5,5,3,3]$-x_2, $[5,5,4,4,3]$-x_3, $[5,4,4,4,4]$-x_4.

By similar reasoning as the last two examples we have:

$x_1 = 8$

$2x_2 + x_3 = 12$

$x_1 + 2x_3 + 4x_4 = 20$

$3x_1 + 3x_2 + 2x_3 + x_4 = 45$

$x_1 + x_2 + x_3 + x_4 = 17$

There are four solutions:

$(8, 4, 6, 0)$

$(8, 4, 4, 1)$

$(8, 5, 2, 2)$

$(8, 6, 0, 3)$

Case 1: $(8, 5, 6, 0)$. There are 9 elements of multiplicty 5, so $\binom{9}{2}$ pairs.

8 blocks of type $[5,5,5,4,2]$ gives $8 \times \binom{3}{2} = 24$

4 blocks of type $[5,5,5,3,3]$ gives $4 \times \binom{3}{2} = 12$

6 blocks of type $[5,5,4,4,3]$ gives $6 \times \binom{2}{2} = 6$

This is $24 + 12 + 6 = 42 > 36$. Contradiction.

Case 2: $(8, 4, 4, 1)$. There are 9 elements of multiplicty 5, so $\binom{9}{2}$ pairs.

8 blocks of type $[5,5,5,4,2]$ gives $8 \times \binom{3}{2} = 24$

4 blocks of type $[5,5,5,3,3]$ gives $4 \times \binom{3}{2} = 12$

4 blocks of type $[5,5,4,4,3]$ gives $4 \times \binom{2}{2} = 4$

This is $24 + 12 + 4 = 40 > 36$. Contradiction.

Case 3: $(8, 5, 2, 2)$. There are 9 elements of multiplicty 5, so $\binom{9}{2}$ pairs.

8 blocks of type $[5,5,5,4,2]$ gives $8 \times \binom{3}{2} = 24$

5 blocks of type $[5,5,5,3,3]$ gives $5 \times \binom{3}{2} = 15$

2 blocks of type $[5,5,4,4,3]$ gives $2 \times \binom{2}{2} = 2$

This is $24 + 15 + 2 = 41 > 36$. Contradiction.

Case 4: $(8, 6, 0, 3)$. There are 9 elements of multiplicty 5, so $\binom{9}{2}$ pairs.

8 blocks of type $[5,5,5,4,2]$ gives $8 \times \binom{3}{2} = 24$

6 blocks of type $[5,5,5,3,3]$ gives $6 \times \binom{3}{2} = 18$

This is $24 + 18 = 42 > 36$. Contradiction.

The examples inspire the following theorem. We leave the proof to the reader.

Definition 3.1 Let $BT = [b_1, b_2, b_3, b_4, b_5)$ be a block type. We need to look at BT this as a multiset and ask how many of each element is in it. For $1 \le i \le 5$ let $m_i(BT)$ be the number of times i is in BT. For example, if $BT = [5, 5, 4, 4, 3]$ then $m_1(BT) = 0$, $m_2(BT) = 0$, $m_3(BT) = 1$, $m_4(BT) = 2$, $m_5(BT) = 2$.

Theorem 3.2 *Let $n, m, n_1, n_2, n_3, n_4, n_5 \in \mathsf{N}$ such that $\sum_{i=1}^{n} n_i = n$. Let $BT_1, \ldots, BT_L$ be all 5-tuples in monotone decreasing order that sum to $m + 4$ (so potential blocktypes). Consider the following 6 linear equations in L variables:*

For $1 \le i \le L$, $1 \le j \le 5$, let t_i^j be the number of j's in T_i. The equations are

$x_1 t_1^1 + x_2 t_2^1 + \cdots + x_L t_L^1 = 1 \times n_1.$

$x_1 t_1^2 + x_2 t_2^2 + \cdots + x_L t_L^2 = 2 \times n_2.$

$x_1 t_1^3 + x_2 t_2^3 + \cdots + x_L t_L^3 = 3 \times n_3.$

$x_1 t_1^4 + x_2 t_2^4 + \cdots + x_L t_L^4 = 4 \times n_4.$

$x_1 t_1^5 + x_2 t_2^5 + \cdots + x_L t_L^5 = 5 \times n_5.$

$x_1 + x_2 + \cdots + x_L = m$

Assume that for every solution $(s_1, \ldots, s_L)$

$$\sum_{j=1}^{L} s_j \binom{m_i(BT_j)}{2} > \binom{n_i}{2}.$$

Then there is no n-awesome collection of m blocks where every $y \in [n]$ occurs ≤ 5 times of type $(n_1, n_2, n_3.n_4, n_5)$.

EIGHTH PROGRAM TO WRITE

We want to, given n, m, $(n_1, n_2, n_3, n_4, n_5)$ form the set of equations similar to those in the example.

Here is a sketch

1. Input $n, m, n_1, n_2, n_3, n_4, n_5$
2. Find the blocktypes $BT_1, \ldots, BT_L$.
3. Use the SEVENTH program to find 6 linear equations in L variables and their solutions. If there are no solutions report that and stop.
4. For each solution $(s_1, \ldots, s_L)$ see if

$$\sum_{j=1}^{L} s_j \binom{m_i(BT_j)}{2} > \binom{n_i}{2}.$$

5. Report on what you find. If for all solutions the inequality holds then report that, and say WOW, NO. If there is a solution for which the inequality does not hold, report that.

END OF EIGHTH PROGRAM TO WRITE