Polynomials Mod 1
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1 Pre Introduction

Everything in this document is a scaled down version of what is in Croot, Lyall, Rice [1].

Notation 1.1 Let $\alpha \in \mathsf{R}$.

1. $\alpha \pmod 1$ is β such that $\beta \in [0,1)$ and $\alpha - \beta \in \mathsf{N}$.

   Examples:

   $10.2 \pmod 1 = 0.2$.

   $\pi \pmod 1 = 0.1415\ldots$

   $10.9 \pmod 1 = 0.9$.

   $e \pmod 1 = 0.7182\ldots$.

2. If $\alpha \in \mathsf{R}$ then $||\alpha||$ is the distance from α to the nearest integer. This is not $\alpha \pmod 1$

   Examples:

   $||10.2|| = 0.2$

   $||\pi|| = 0.1415\ldots$.

   $||10.9|| = 0.1$

   $||e|| = 0.2817\ldots$.

3. $a \ll b$ means that a is less than a constant times b.

2 Introduction

The following theorem is a special case of the Kronecker Approx Theorem. We give a purely combinatorial proof that was due to Kronecker.

Theorem 2.1 *Let $\alpha \in \mathsf{R}$ and $N \in \mathsf{N}$, $N \geq 1$. Then there exists $1 \leq n \leq N$ such that*

$$||n\alpha|| \ll \frac{1}{N}$$

Proof: View $[0,1]$ as

$$\left[0, \frac{1}{N-1}\right) \bigcup \left[\frac{1}{N-1}, \frac{2}{N-1}\right) \bigcup \cdots \bigcup \left[\frac{N-2}{N-1}, 1\right).$$

Map each $1 \leq i \leq N$ to the interval that $i\alpha \pmod 1$ is in. We are mapping N numbers to $N-1$ intervals, hence two of them map to the same interval. Let them be $i\alpha$ and $j\alpha$. Hence
$i\alpha = m_i + \epsilon_i$
$j\alpha = m_j + \epsilon_j$