# BILL, RECORD LECTURE!!!!

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# Rado's Thm

**Exposition by William Gasarch**

April 29, 2025

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What about other equations?

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(We can modify the proof to get a d-mono sol.)

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**Thm** $x + y = z$ is regular. (Can also show d-regular.)

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Take $x = y = z = 1$. Or any $x = y = z$.

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So get $w = a \qquad x = a + 5d \qquad y = a + 3d$.

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What is it about

$$2w + 3x = 5y$$

that made all of the $a$'s drop out? Discuss.

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We won't prove this but you have seen most of the ideas needed to prove it.

# Other Equations

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We won't prove. You have seen most of the ideas needed to prove it.

# An Equation Where Rado Fails

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$3b \equiv 0 \pmod 5$ implies $b \equiv 0 \pmod 5$. Contradiction.

# Case $e_1 = e_3 < e_2$ and $e_2 = e_3 < e_1$

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Take this mod 5 to get $3b \equiv 4b$ so $b \equiv 0 \pmod 5$ Contradiction.

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   2.1 We need a prime $p$
   2.2 We needed $3b \equiv 0 \pmod{p}$ implies $b \equiv 0 \pmod{p}$

# What Was Special About $x + 2y = 4z$? About 5?

1. The proof used that NO subset of $1, 2, -4$ sums to 0.
2. We used 5 since
   2.1 We need a prime $p$
   2.2 We needed $3b \equiv 0 \pmod{p}$ implies $b \equiv 0 \pmod{p}$
   5 is the lowest such prime.

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**Research Question**

1. For $x + 2y = 4z$ what about 4-coloring? 3-coloring? 2-coloring?
2. More generally one can take an equation where no sum of the coefficients is 0 and look at colorings with a small number of colors.

# Full Rado

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(For most equations with the coefficients sum to 0 you actually get d-regular.)

# Misc

# Research Questions

(Some is known about some of these.)
Prove or disprove that the equations below are regular.

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2. Higher degree equations (seems hard).

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   (This equation has certain properties that make it work, so there is really a more general thm here.) `http://fourier.math.uoc.gr/~ergodic/Slides/Host.pdf`

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3) (Might be Hard) Obtain a human-readable proof with perhaps a much bigger $N$, but which can be generalized to $c = 3$ and beyond.

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The SAT Solver is described here:
`https://arxiv.org/pdf/1711.08076`