The First Anti-Ramsey Theorem
An Exposition by
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1 Introduction

Definition 1.1 If $n \in \mathsf{N}$ then $[n] = \{1, \ldots, n\}$.

The following is Schur's Theorem which is an early result (1916) in Ramsey theory:

Theorem 1.2 *For all $c \in \mathsf{N}$ there exists $n \in \mathsf{N}$ such that, for all* COL$\colon [n] \to [c]$ *there exists x, y, z all the same color such that $x + y = z$.*

In Schur's theorem the goal is a solution to $x + y = z$ where x, y, z are *the same* color. What if you want a solution where x, y, z are *different colors*?

Definition 1.3 Let $c, n \in \mathsf{N}$ and COL$\colon [n] \to [c]$. A triple $x, y, z \in [n]$ is *rainbow* if x, y, z are all different colors.

If COL colors all the numbers R then there is no rainbow solution. Our main theorem deals with this issue.

2 The Main Theorem

The following theorem was proven in 1987 by NEED AUTHORS NAMES.

Theorem 2.1 *Let $n \in \mathsf{N}$. Let* COL$\colon [3n] \to [3]$. *Assume that every color appears in the image n times. Then there exists rainbow $x, y, z \in [3n]$ such $x + y = z$*

Proof:
We let the colors be R,B,G. We can assume COL$(1) = $ R. Let k be such that

$$\text{COL}(1) = \cdots = \text{COL}(k-1) \neq \text{COL}(k).$$

We can assume COL$(k) = $ B.
We do an example: $n = 7$, $k = 5$, and COL$(13) = $ G. (When we generalize the example we will have a instead of 13). So we have:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
R	R	R	R	B								G								

We will show that either we get our rainbow solution or COL$(12) = $ R. When we generalize this example (1) we will have $a - 1$ instead of 12, and (2) we will see how getting COL$(a-1) = R$ helps prove the theorem.
Case 1: COL$(12) = $ B. So we have:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
R	R	R	R	B							B	G								

Then $(1, 12, 13)$ is a rainbow solution. (In generalization, 12 is replaced with $a - 1$.)

Case 2: COL(12) = G. So we have:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
R	R	R	R	B							G	G								

Case 2.1: COL(8) = R. So we have:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
R	R	R	R	B			R				G	G								

Then $(5, 8, 13)$ is a rainbow solution. (This case did not use COL(12) = G.) (In generalization, 8 is replaced with $a - k$.)

Case 2.2: COL(8) = B. So we have:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
R	R	R	R	B			B				G	G								

Then $(4, 8, 12)$ is a rainbow solution.

Case 2.3: COL(8) = G. (We will go through cases that look odd; however, they are similar to what happens when we generalize this example.) So we have:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
R	R	R	R	B			G				G	G								

Case 2.3.1: COL(3) = B. NOT TRUE. (When we generalize this example we will have $a - 2k$ instead of 3.)

Case 2.3.2: COL(3) = R. TRUE. Rainbow solution $(2, 5, 8)$.

Case 2.3.2.1: COL(3) = G. NOT TRUE.

Case 3: COL(12) = R. This must be what happens.

We generalize this example.

Case 1: $\exists a \geq k + 1$, COL(a) = G and COL(a − 1) = B. Then $(1, a − 1, a)$ is a rainbow solution.

Case 2: $\exists a \geq k + 1$, COL(a) = G and COL(a − 1) = G.

Case 2.1: COL(a − k) = R. ($a − k \in [n]$ since $a \geq k + 1$.)

Then $(k, a − k, a)$ is a rainbow solution. (This case did not use that COL(a − 1) = G.)

Case 2.2: COL(a − k) = B.

Then $(k − 1, a − k, a − 1)$ is a rainbow solution.

Case 2.3: COL(a−k) = G. Replicate the reasoning from Case 1,2,2.1,2.2 with a replaced by $a−k$. There is either a rainbow solution or COL(a − 2k) = G. If COL(a − 2k) = G then (1) $a − 2k \in [k]$: contradiction, or (2) repeat the argument again. Keep doing this. Eventually there is a rainbow solution since otherwise $\exists i$, $a − ik \in [k]$ and hence cannot be green.

Case 3: $\forall a$ if COL(a) = G then COL(a − 1) = R. We map every green number a to $a − 1$. This is an injection and everything in the image is red. The number 1 is not in the image since either COL(2) = R or COL(2) = B. Hence there is at least one more green number than red number. But the number of green numbers is the same as the number of red numbers (both numbers are n). Hence this case cannot occur.

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