Homework 05, Morally Due 12:30PM, Tue Mar 03 2026

1. (0 points) What is your name?

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2. (35 points) Let $\Sigma = \{a, b\}$. Let Σ^* be the set of all strings over Σ including the empty string (Example: *aabba*, *bababbba*.)

   If $x, y \in \Sigma^*$ then x is a *subsequence of* y if you can take y, remove some letters, and get x.

   Example: *abba* is a subsequence of *aababaabaaaabaa*

   We define $x \preceq y$ to mean that x is a subsequence of y.

   Show that $(\Sigma^*, \preceq)$ is a well quasi order by using a minimal bad sequence argument.

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3. (0 point but you must do this to appreciate what I tell you when I go over the subsequence problem in class. Do not hand anything in for this problem.)

   If $w \in \Sigma^*$ then SUBSEQ(w) is the set of all subsequences of w.

   Let $L \subseteq \{a, b\}^*$. Then SUBSEQ(L) $= \cup_{w \in L}$SUBSEQ(w).

   (a) Look at these two slide packets from my CMSC 452, on regular languages.
   `https://www.cs.umd.edu/~gasarch/COURSES/452/S25/slides/dfatalk.pdf`
   `https://www.cs.umd.edu/~gasarch/COURSES/452/S25/slides/nfatalk.pdf`
   Using the DFA NFA equivalence show the following:

   If L is regular then SUBSEQ(L) *is regular.*

   (b) Look at this two slide packets from my CMSC 452, on context free languages. (Only need up to around slide 23—the definitions of a context free grammar and of a context free language.)
   https://www.cs.umd.edu/~gasarch/COURSES/452/S25/slides/cfgtalk.pdf
   Using the definition of context-free grammar show the following:

   If L is a context-free language then SUBSEQ(L) *is a context-free language.*

   (c) I assume you all know what P is, the set of languages in polynomial time.
   Is the following TRUE, FALSE, or UNKNOWN TO SCIENCE:

   If L is in P then SUBSEQ(L) *is in P.*

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4. (35 points) For this problem I define the minor ordering on colored graphs. $H \preceq_{\text{c-m}} G$, as follows: $H \preceq_{\text{c-m}} G$ if you can take G and, by a sequence of the following operations, obtain H: (a) contract an edge and choose one of the colors of the endpoints to be the color of the merged vertex, (b) remove a vertex, (c) remove an edge.

   For this problem you can assume the GMT for COLORED graphs:

   Let $n \in \mathsf{N}$. If $G_1, G_2, \ldots,$ is an infinite sequence of n-colored graphs then there exists $i < j$ such that $G_i \preceq_{\text{c-m}} G_j$.

   Let GM(n) be the largest number such that there exists a bad sequence (using $\preceq_{\text{c-m}}$)) of n-colored graphs

   $G_1, G_2, \ldots, G_{\text{GM}(n)}$

   where G_i has at most i vertices.

   Show that GM(n) exists.

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5. (30 points) Fix $k \in \mathsf{N}$. Assume that, for all $1 \le i \le k$, WE HAVE an FPT algorithm for $VC_i = \{G \colon G \text{ has a vertex cover of size } \le i\}$.
(Recall that this means we really have the code.)
Prove that WE HAVE an FPT algorithm for the following FUNCTION:
On input $G = (V, E)$:

- If $G \notin VC_k$ then output NO.
- If $G \in VC_k$ then output YES and ALSO output a vertex cover $U \subseteq V$ of size $\le k$.

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6. (0 points. Do for your own enlightenment. DO NOT hand in.)

   Let $(X, \preceq)$ be a wqo. $2^{\text{fin}X}$ is the set of finite subsets of X.

   We define the following ordering on $2^{\text{fin}X}$

   $A \preceq_1 B$ if there exists a 1-1 function $f \colon A \rightarrow B$ such that $x \preceq f(x)$.

   Note that $\preceq_1$ is defined using $\preceq$.

   Show that $(2^{\text{fin}X}, \preceq_1)$ is a wqo.

   Hint: This is an adjustment of the proof that $2^{\text{fin}\mathsf{N}}$ under a similar ordering is a wqo.

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7. (Extra Credit)

I first discuss the subtlety about the Kruskal Tree Theorem that I alluded to in class.

I defined the minor ordering, denoted $H \preceq_m G$, as follows: $H \preceq_m G$ if you can take G and, by a sequence of the following operations, obtain H: (a) contract an edge. (b) remove a vertex, (c) remove an edge.

Here is what I stated in class which, while true, is not what you should try to prove:

STATEMENT A: *The set of trees under* $\preceq_m$ *is a wqo.*

DO NOT try to prove STATEMENT A. It is true; however, to prove it you need to prove something *stronger*. This is one of those cases where it is easier to prove a stronger theorem.

I now define the minor' ordering, denoted $H \preceq'_m G$, as follows: $H \preceq'_m G$ if you can take G and, by a sequence of the following operations, obtain H: (a) contract an edge. Oh. Just one operation.

Here is what you can prove similar to the other proofs:

STATEMENT B: *The set of trees under* $\preceq'_m$ *is a wqo.*

For the extra credit prove STATEMENT B. ALSO (and this is probably the only part I will look at)

(a) State clearly which part of the proof breaks down if you use $\preceq_m$ instead of $\preceq'_m$.

(b) State clearly which part of the proof breaks down if you use SUBGRAPH instead of $\preceq'_m$.