**Homework 06, Morally Due 12:30PM, Tue Mar 10 2026**

1. (0 points) What is your name?

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2. (35 points) Recall that we proved the following statement:

   *For all* COL: $\binom{\mathsf{N}}{2} \rightarrow [2]$ *there exists a homog set.*

   FROM this give a nonconstructive proof of the following:

   *For all* $k$ *there exists* $n$ *such that for all* COL: $\binom{[n]}{2} \rightarrow [2]$ *there exists a homog set of size* $k$.

   The proof should begin with

   *Assume not. Then there exists* $k$ *such that, for all* $n$, *there exists* $\mathrm{COL}_n \colon \binom{[n]}{2} \rightarrow [2]$ *with no homog set of size* $k$.

   **Hint:** The proof is similar to the proof that tree($n$) exists.

   **Warning:** If you go to the web or AI you will probably find a CONSTRUCTIVE proof that $n$ exists and $n \le 2^{2k}$ or even lower. This is NOT what I want. (I will do that proof later in the term.)

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3. (35 points) Consider the following algorithm for $\mathrm{VC}_k$:

**BEGIN**

ALG4$(G, k)$ (Note that we call it ALG4. We use this terminology later.)

Input $G, k$

If $k = 0$ and $G$ has at least one edge then output NO. This take $n$ steps.

Look for a vertex of degree $\geq 3$.

- If none exists then every vertex has degree $\leq 2$. This graph is just a collection of lines and cycles. Finding if there is a VC of size $k$ takes $n$ steps.
- There exists a vertex $v$ of degree $\geq 3$. Consider the two cases of either USING $v$ IN THE VC or NOT USING $v$ IN THE VC.
  - Use $v$. If ALG4$(G - \{v\}, k - 1)$ says YES then output YES.
  - Do not use $v$. *Then you must use all of its neighbors!*. Let $N(v)$ be the neighbors of $v$. If ALG4$(G - (N(v) \cup \{v\}), k - |N(v)|)$ says YES then output YES.
  - Output NO.

**END**

(a) (15 points) Write a recurrence that bounds $T(n, k)$. For example it could be (but it's NOT)

$T(n, 0) = \mathrm{ACK}(n, n)$.

$T(n, k) \leq T(n - \lceil \log n \rceil, k - \lceil \sqrt{k} \rceil) + k2^n$.

(b) (0 points but you need to do this.) Write a program that computes an upper bound on $T(n, k)$ using Part $a$. (You do not hand anything in for this part.)

*Note* You need to have done Part $a$ to do this part. If you are having trouble with Part $a1$ you have permission (more than usual) to get help on it since I really want you to do Part $c$.

(c) (10 points) We think that $T(n, k)$ is bounded by an expression of the form $\alpha^k n$.

Do empirical studies to find a good value for $\alpha$.

Give a good value for $\alpha$ and describe why your empirical evidence gives you that value.

(d) (10 points) Consider the following algorithm: Run Buss's algorithm except, at the last step, instead of using ALG2, use ALG4. What is the run time? Express in terms of $\alpha, k, n$ and you can use O-of (so it could be, but its not $O(\mathrm{ACK}(\alpha, k)k^n n^k)$.

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4. (30 points) In class I STATED that the following are equivalent: Axiom of Choice (AC), Zorn's Lemma (ZL), and The Well Ordering Principle (WOP).

   Prove one of the following (you can go to the web or AI for help but you must put it in your own words).

   $AC \implies ZL$

   $ZL \implies AC$

   $AC \implies WOP$

   $WOP \implies AC$

   $ZL \implies WOP$

   $WOP \implies ZL$

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5. (Extra Credit- 0 points) Give a analytic proof of a good bound on $T(n,k)$ in Problem 2. The proof should be complete and well written so that I can easily make it into slides and present it to the class. (I am not sure if I will do this, but I want to be able to.)