

# BILL, RECORD LECTURE!!!!

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# One Triangle, Two Triangles

**William Gasarch**

# Lets Party Like Its 2019

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The following is the first theorem in Ramsey Theory:

**Thm** For all 2-col of the edges of  $K_6$  there is a mono  $K_3$ .

# Trivial Theorem, Non Trivial Extension

**Thm** For all 2-cols of edges of  $K_{12}$  there are 2 mono  $K_3$ 's

**Question** Find  $n$  such that

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# Proof of $K_6$ Two Triangles Theorem

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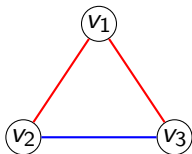
**Proof** Let  $COL$  be a 2-col of the edges of  $K_6$ .

Let  $R$ ,  $B$ ,  $M$ , be the SET of **RED**, **BLUE**, and **MIXED** triangles.

$$|R| + |B| + |M| = \binom{6}{3} = 20.$$

We show that  $|M| \leq 18$ , so  $|R| + |B| \geq 2$ .

## A Mixed Triangle Has a Vertex Such That



- ▶  $(v_2, v_1)$  is red,  $(v_2, v_3)$  is blue. View this as  $(v_2, \{v_1, v_3\})$ .
- ▶  $(v_3, v_1)$  is red,  $(v_3, v_2)$  is blue. View this as  $(v_3, \{v_1, v_2\})$ .

## Map ZAN to $M$

**Def** A **Zan** is an element  $(v, \{u, w\}) \in V \times \binom{V}{2}$  such that  $v \notin \{u, w\}$  and  $COL(v, u) \neq COL(v, w)$ . ZAN is the set of Zan's.

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Map ZAN to  $M$  by mapping  $(v, \{u, w\})$  to triangle  $\{v, u, w\}$ .

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**Claim** This mapping is exactly 2-to-1.

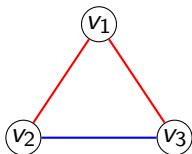
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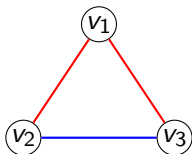
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$(v_2, \{v_1, v_3\})$  and  $(v_3, \{v_1, v_2\})$ .

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So  $v$  contributes  $\deg_R(v) \times \deg_B(v)$ .

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So there are at least 2 Mono Triangles.

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We find an upper bound on  $|ZAN|$ .

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$$\begin{aligned} |R| + |B| &\geq \frac{n(n-1)(n-2)}{6} - \frac{(n-1)^2 n}{8} \\ &= \frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24} \end{aligned}$$

# What About The Other Cases?

We leave the other cases to the reader to both determine the theorem and prove it.

# Can This Be Improved?

The bound is known to be tight.

# Coloring that Shows the Bound is Tight

Go to this pointer: `tttalk.pdf`

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The ratio  $\frac{1}{8}$  is tight.

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2. I will present a math-interesting proof of the following:  
*For all 2-cols of  $K_{19}$  there are TWO mono  $K_4$ 's.*

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For the real proof, see next slide.

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Those are our 2 mono  $K_4$ 's.

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**Note** We only showed that the **proof** cannot be extended. As noted above any 2-col of  $K_{18}$  has 9 mono  $K_4$ 's.

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 $\binom{n-4}{18-4} = \binom{n-4}{14}$  of these. There are  $\binom{n}{18} - \binom{n-4}{14}$  left.
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## Want $n$ such that $\forall$ 2-col $\exists$ 3 Mono $K_4$ 's

$\forall$  2-col of  $K_n \exists$  3 mono  $K_4$ 's.

List out all subsets of  $V = \{1, \dots, n\}$  of size  $R(4) = 18$ .

$$A_1, A_2, \dots, A_{\binom{n}{18}}.$$

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## Want 3 Mono $K_4$ 's (cont)

Need

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## Want 3 Mono $K_4$ 's (cont)

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## Want 3 Mono $K_4$ 's (cont)

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$n$	$n(n-1)(n-2)(n-3)$
19	93024
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**Thm**  $\forall$  2-cols of the edges of  $K_{22} \ni$  3 mono  $K_4$ 's.

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Since  $\binom{n}{18} - m\binom{n-4}{14} \geq 1$  this process can go for  $\geq m + 1$  iterations and produce  $\geq m + 1$  mono  $K_4$ 's.

## Want $m$ Mono $K_4$ 's (cont)

We just proved that for all  $n, m \in \mathbb{N}$ :

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We state a theorem which expresses this in several ways, on the next slide.

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4) There are  $\frac{n^4}{73440} - \frac{n^3}{12240} + \Omega(n^2)$  mono  $K_4$ 's.

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Is better known? Speculate. Answer on next page.

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This will be a HW.

# Generalize

Left to the reader

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3. Generalize to  $c$  colors and mono  $K_m$ .