BILL, RECORD LECTURE!!!!

BILL RECORD LECTURE!!!

Convex Points Thm
Known as
Happy Ending Thm

Exposition by William Gasarch

Convex Sets
And
Convex Hulls

Convex Sets

Convex Sets

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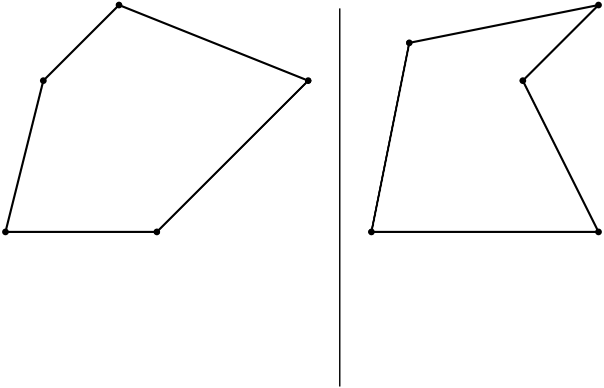
Convex and Non-Convex Sets on Next Slide.

Convex Set / Non-Convex Set



Left Region is Convex. Right Region is Not Convex.

Definition of A Convex Hull

Def Let $X \subseteq \mathbb{R}^2$ (it will always be a finite set). The **Convex Hull of** X is the smallest convex set that contains all the points in X.

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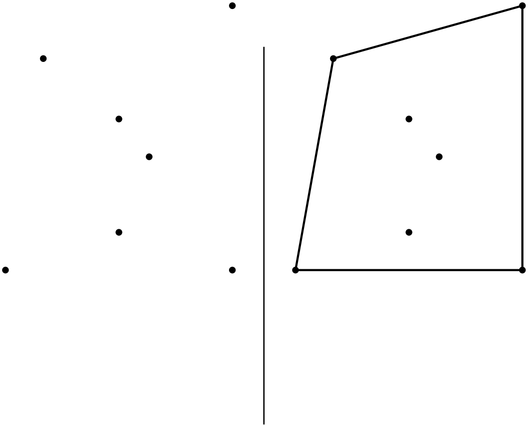
An Example is on Next Slide.

Size of a Convex Hull

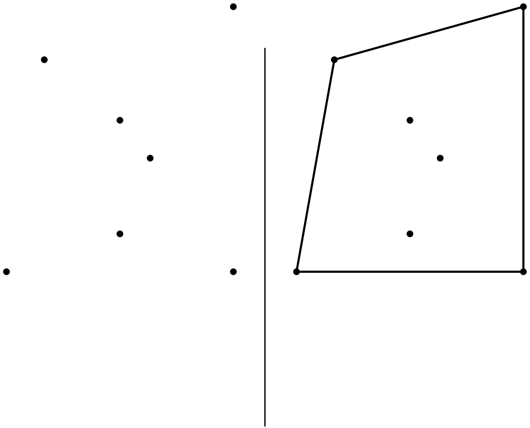
Def The **Size of a Convex Hull** is how many sides it has.

Example of A Convex Hull



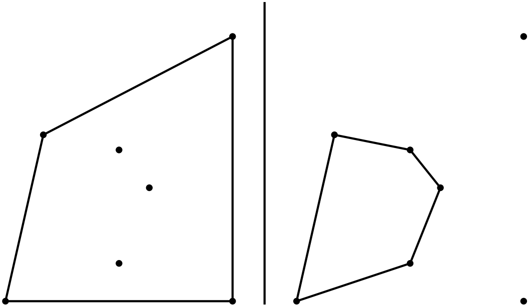
Region In Right Picture is Convex Hull of Points in Left Picture.

Example of A Convex Hull



Region In Right Picture is Convex Hull of Points in Left Picture.
RHS is a convex hull of size 4.

Convex Hull Not the Largest Convex Hull



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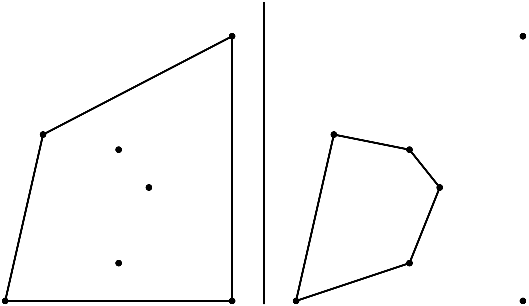

LHS: 7 points have a convex hull of size 4.

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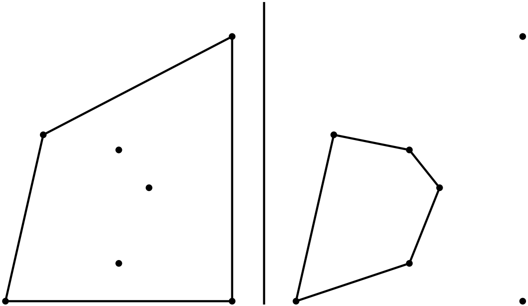

LHS: 7 points have a convex hull of size 4.

RHS: 5 of those 7 point have a convex hull of size 5.

We Want Large Convex Hulls
Concrete Examples

Given n Points in $\mathbb{R}^2$ Want Large Convex Hull

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Answer on Next Slides.

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Nothing is known beyond that but note that fits $f(k) = 2^{k-2} + 1$ conjecture.

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Hence $f(4) \geq 5$.

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We do example on the next page.

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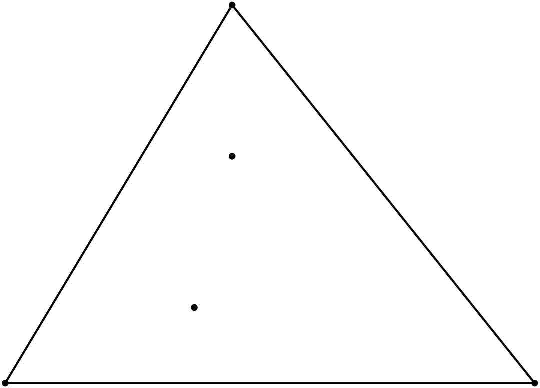
General proof left to the reader.

Two Points Inside a 3-gon



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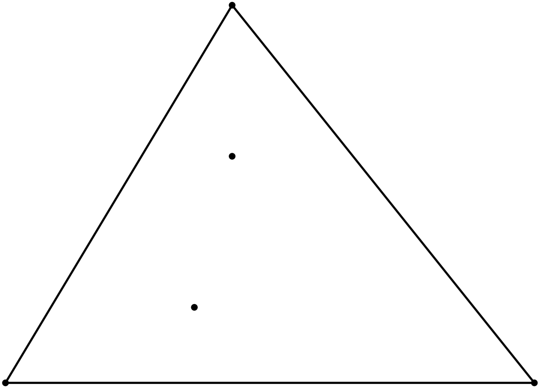
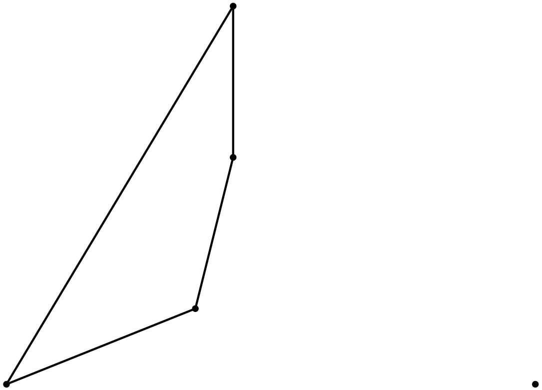


See Next Slide for the Amazing 4-Gon!

The Amazing 4-Gon



We Want Large Convex Hulls For General k

Can We Always Get k Sized Convex Hull

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See next slide for the exciting conclusion to this story.

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Erdös & Szekeres proved: $(\forall k)(\exists n)(\forall X \subseteq \mathbb{R}^2, |X| = n)(\exists Y \subseteq X)$ $|Y| = k$, and the convex hull of Y is a k-gon.

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Ryan's Friend You're going to prove they were married 68 years and died on the same day 3 ways?

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Erdös & Szekeres proved: $(\forall k)(\exists n)(\forall X \subseteq \mathbb{R}^2, |X| = n)(\exists Y \subseteq X)$ $|Y| = k$, and the convex hull of Y is a k-gon.

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I am kidding of course.

Proof Using $R_4(5, k)$

This proof is by Erdös-Szekeres

Recall Let $n = R_4(5, k)$. Then for all $\mathrm{COL}\colon \binom{[n]}{4} \rightarrow [2]$ there is RED homog set of size 5 OR a BLUE homog set of size k.

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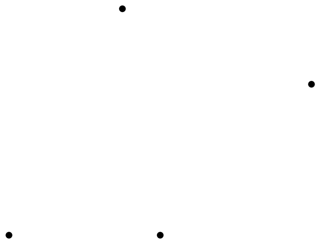
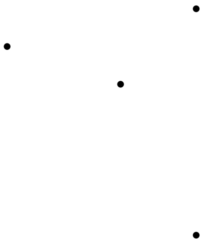
Let $Z \in \binom{X}{4}$.

$$\mathrm{COL}(Z) = \begin{cases} \mathrm{CONV} \text{ if } Z \text{ forms a convex quadrilateral} \\ \mathrm{NOTCONV} \text{ otherwise} \end{cases} \tag{1}$$

Example Of the Coloring



Colored CONV.



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2) $|H| = k$ with COL restricted to $\binom{H}{4}$ is the constant CONV.
Exercise Show that if every 4-subset of H is a convex 4-gon then H is a convex hull of size k.

Proof Using $R_3(k,k)$ (by P. Johnson (1986))

Recall Let $n = R_3(k,k)$. Then for all $\mathrm{COL}\colon \binom{[n]}{3} \rightarrow [2]$ there is a **R** homog set of size k OR a **B** homog set of size k.

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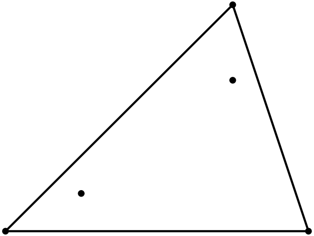
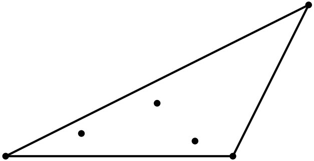
Example of The Coloring



Colored EVEN.



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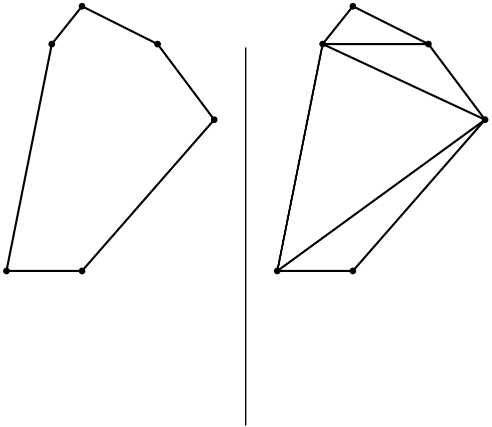
So every 3-subset of H has an odd number of points inside.
We Proof there are no points of H in the convex hull of H and hence we have a convex hull of size k.

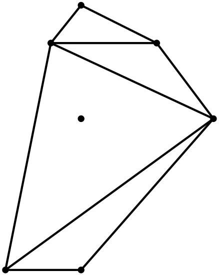
Convex Hull of *H* and its Triangulation



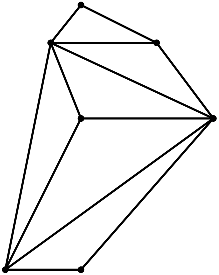
Convex Hull of H. Triangulation.

We need to prove that there are no points of H in the convex hull.

Assume There is a Point of *H* in the Convex Hull



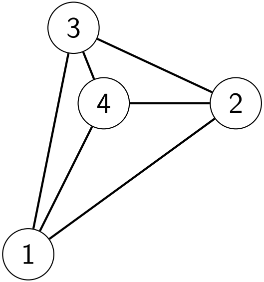
Which △ point is in.



More Triangulation.

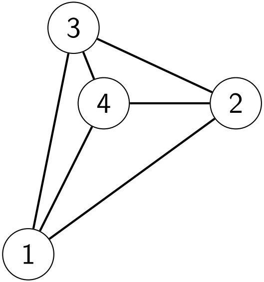
Need just the new point and its neighbors, and need labels.

Parity Argument



All these points are in H.

Parity Argument



All these points are in H. Assume all $\triangle$ colored EVEN.

$1 - 2 - 4$ has an even number of points

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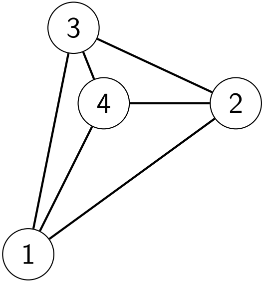

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Not possible. $1 - 2 - 3$ has the points of $1 - 2 - 4$ AND $2 - 3 - 4$ AND $1 - 3 - 4$ AND the point 4. Thats Odd!

Third Proof

The Third Proof will be on the HW

A Bit More History

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The proof that Erdös-Szekeres had was the $R_4(5, k)$.

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Calling the proof **an application of Ramsey Theory** is a bit odd since it is really **one of the two reasons Ramsey Theory was invented** (the other was Ramsey's problem in logic).