A 2-Coloring of R^2 with no Red L_3 or Blue $L_{\rm big}$
Exposition by William Gasarch, Chaewoon Kyoung, Kelin Zhu

1 Pre Introduction

The following is well known.

Theorem 1.1 *For all* COL: $\mathsf{R}^2 \rightarrow [2]$ *there exists 2 points, same color, 1 inch apart.*

We rephrase this but first need some definitions.

Definition 1.2

1. ℓ_2 is 2 points in the plane an inch apart.
2. ℓ_3 is three colinear points p_1, p_2, p_3 where $d(p_1, p_2) = d(p_2, p_3) = 1$.
3. You can define ℓ_k.
4. Given COL: $\mathsf{R}^2 \rightarrow [2]$, a *RED* ℓ_k is an ℓ_k where all the points in it are RED. Similar for a BLUE ℓ_k.

Notation 1.3 Let $n, a, b \geq 2$. $\mathsf{R}^n \rightarrow (\ell_a, \ell_b)$ means that, for all COL: $\mathsf{R}^n \rightarrow [2]$, either there is a RED ℓ_a or a BLUE ℓ_b.

Many results are known about when $\mathsf{R}^n \rightarrow (\ell_a, \ell_b)$ and when $\mathsf{R}^n \not\rightarrow (\ell_a, \ell_b)$. We *do not* summarize them here. (When this document becomes part of a larger document we will.)

Conlon & Wu [2] showed that there exists m (around 10^{50}) such that

$$(\forall n)[\mathsf{R}^n \not\rightarrow (\ell_3, \ell_m)].$$

Implicit in their proof was a result in (what we call) Rado's theorem over the reals. They did not present it that way, nor did they isolate it from the rest of the proof. They also only proved the case that they needed.

In this document we present a generalization of that result and then prove the Conlon-Wu result.

2 Theorems About Intersection

The definition of $a \pmod q$ is well known if $a \in \mathsf{Z}$ and $q \in \mathsf{N}^{\geq 2}$. We need to define this concept for when $a \in \mathsf{R}$.

Definition 2.1

1. Let $a \in \mathsf{R}$ and $q \in \mathsf{N}^{\geq}2$. Then $a \pmod q$ is the unique a' such that (1) $a - a'$ is an integer multiple of q (2) $a' \in [0, q)$.

2. Let $f\colon \mathsf{R} \rightarrow \mathsf{R}$. Let $q \in \mathsf{N}$. Then $f \pmod{q}\colon \mathsf{R} \rightarrow [0,q)$ is the function that, on input x, returns the element of

$$\{f(x) + kq\colon k \in \mathsf{Z}\}$$

that is in $[0,q)$.

Example 2.2

1. $\pi \equiv 1.1415\ldots \pmod{2}$.

   $\pi \equiv 2.1415\ldots \pmod{3}$.

   $\pi \equiv \pi \pmod{4}$.

2. $\sqrt{205} \sim 14.41782\ldots$. Hence

   $\sqrt{205} \equiv 1.41782\ldots \pmod{2}$.

   $\sqrt{205} \equiv 2.41782\ldots \pmod{3}$.

   $\sqrt{205} \equiv 3.41782\ldots \pmod{4}$.

   $\vdots$

   $\sqrt{205} \equiv 13.41782\ldots \pmod{14}$.

3. Let $f(x) = x^2 + \pi x + e \pmod{13}$

   Then

   $f(10) = 100 + 10\pi + e \pmod{13} \sim\sim 4.1324$.

Note 2.3 Let $f \in \mathsf{R}[x]$. Let q be a prime. All $\equiv$ are mod q. We wonder when the following is true:

$$(1) \qquad a \equiv b \implies f(a) \equiv f(b).$$

We consider four cases by looking at (1) $a, b \in \mathsf{Z}$ or $a, b \in \mathsf{R}$, and (2) $f \in \mathsf{Z}[x]$ or $f \in \mathsf{R}[x]$.

1. If $a, b \in \mathsf{Z}$ and $f \in \mathsf{Z}[x]$ then (1) is TRUE.

2. If $a, b \in \mathsf{Z}$ and $f \in \mathsf{R}[x]$ then (1) is FALSE. Let $q = 13$, $a = 10$, $b = 23$, $f(x) = 0.5x$.

   $f(10) \bmod 13 = 5 \bmod 13 = 5$

   $f(23) \bmod 13 = 11.5 \bmod 13 = 11.5$

3. If $a, b \in \mathsf{R}$ and $f \in \mathsf{Z}[x]$ then (1) is FALSE. Let $q = 13$, $a = 13 + \frac{1}{13}$, $b = 130 + \frac{1}{13}$, and $f(x) = x^2$

   $f(13 + \frac{1}{13}) = 13^2 + 2 + \frac{1}{169} \equiv 2 + \frac{1}{169} \sim 2.005617$

   $f(130 + \frac{1}{13}) = 130^2 + 20 + \frac{1}{169} \equiv 7 + \frac{1}{169} \sim 7.005617$

4. If $a, b \in \mathsf{R}$ and $f \in \mathsf{R}[x]$ then (1) is FALSE. Either the second or third example on this list suffices.

Consider again

$$f(x) = x^2 + \pi x + e \pmod{13}$$

Let $m \in \mathsf{N}$ (we are thinking of m large). Each element of

$$X = \{f(1), f(2), \ldots, f(m)\}$$

is in one of $[0,1)$, $[1,2)$, …, $[12,13)$. We wonder how the elements of X are distributed in those intervals. For example, how many of the intervals $[0,1), [1,2), \ldots, [12,13)$ intersect X.

More generally, Let

1. $\alpha, \beta \in \mathsf{R}$
2. q be a prime.
3. $f(x) = x^2 + \alpha x + \beta \pmod{q}$.
4. $m \in \mathsf{N}$. We think of m has large.
5. $X = \{f(1), f(2), \ldots, f(m)\}$.

Every element of X is in in one of $[0,1)$, $[1,2)$, …, $[q-1,q)$. We wonder how many of the intervals $[0,1), [1,2), \ldots, [q-1,q)$ intersect X. Is it possible that most of the elements of X are in just a few intervals? In the appendix we have some empirical results on this question. The next lemmas answers the question for quadratics over Z, Q, and R. The answer is NO.

Theorem 2.4 *Let $\alpha \in \mathsf{Z}$ and let q be a prime.*

1. *Let $f(x) = x^2 + \alpha x \pmod{q}$.*

   Let

   $$X = \{f(1), f(2), \ldots, f(q)\}.$$

   Then at least $\frac{q+1}{2}$ of the intervals $[0,1)$, $[1,2)$, …, $[q-1,q)$ intersect with X. (Since $a \in \mathsf{Z}$ in reality $f(x) \in \{0, 1, \ldots, q-1\}$. We use intervals to be consistent with the versions over Q and R.)

2. *Let $\beta \in \mathsf{R}$. Let $f(x) = x^2 + \alpha x + \beta \pmod{q}$. Let*

   $$X = \{f(1), f(2), \ldots, f(q)\}.$$

   Then at least $\frac{q+1}{2}$ of the intervals $[0,1)$, $[1,2)$, …, $[q-1,q)$ intersect with X. (Part 2 follows from Part 1.)

Proof:

We show that f is a $\leq$ 2-to-1 map.

Assume $1 \leq i, j \leq q$. All $\equiv$ are mod q.

$$f(i) = f(j)$$

$$i^2 + \alpha i \equiv j^2 + \alpha j$$

$$i^2 - j^2 \equiv \alpha j - \alpha i$$

$$(i - j)(i + j) \equiv \alpha(j - i)$$

Since $i \neq j$ we can divide by $i - j$.

$$(i + j) \equiv -\alpha$$

$$j \equiv -i - \alpha.$$

Hence, for all i, there is at most one j such that $f(i) = f(j)$. Therefore the function f is $\leq$ 2-to-1. So

$$|X| = |\{f(1), \ldots, f(q)\}| \geq \frac{q+1}{2}.$$

Since $f(1), \ldots, f(q) \in \{0, 1, \ldots, q-1\}$, X hits at least $\frac{q+1}{2}$ intervals. (In this case talking about intervals is silly; however, we want this theorem to be similar to the next two.)

∎

Theorem 2.5 *Let $r, s \in \mathsf{Z}$ such that* $\gcd(r, s) = 1$*, q a prime such that $s \not\equiv 0 \pmod{q}$, and $m = qs$.*

1. *Let $f(x) = x^2 + \frac{r}{s}x \pmod{q}$.*

   Let

   $$X = \{f(1), f(2), \ldots, f(m)\}.$$

   Then at least $(q + 1)/2$ of the intervals $[0, 1)$, $[1, 2)$, $\ldots$, $[q - 1, q)$ intersect with X.

2. *Let $\beta \in \mathsf{R}$. Let $p(x) = x^2 + \frac{r}{s}x + \beta$.*

   Let $f(x) = p(x) \pmod{q}$, and

   $$X = \{f(1), f(2), \ldots, f(m)\}.$$

   Then at least $(q + 1)/2$ of the intervals $[0, 1)$, $[1, 2)$, $\ldots$, $[q - 1, q)$ intersect with X. (Part 2 follows from Part 1.)

Proof:

We show that f is $\leq$ 2-to-1 when restricted to

$$X_1 = \{f(s), f(2s), f(3s), \ldots f(qs)\}.$$

Assume $1 \leq i, j \leq q$. All $\equiv$ are mod q.

$$f(is) = f(js)$$

$$(is)^2 + \frac{r}{s} is \equiv (js)^2 + \frac{r}{s} js$$

$$i^2 s^2 + ir \equiv j^2 s^2 + jr$$

$$(i^2 - j^2)s^2 \equiv (j - i)r$$

$$(i - j)(i + j)s^2 \equiv (j - i)r$$

Since $i \neq j$ we can divide by $i - j$.

$$(i + j)s^2 \equiv -r \pmod{q}$$

$$(i + j) \equiv -\frac{r}{s^2}$$

$$i = -j - \frac{r}{s^2}.$$

Hence, for every i, there is at most one $j \neq i$ such that $f(si) = f(sj)$.
Therefore

$$|X| \geq |\{f(s), f(2s), \ldots, f(qs)\}| \geq \frac{q+1}{2}.$$

Since $f(s), f(2s), \ldots, f(qs) \in \{0, \ldots, q-1\}$, X hits at least $\frac{q+1}{2}$ intervals. ▮

Theorem 2.6 *Let $\alpha \in \mathsf{R}$, q a prime, and $m = q^3$.*

1. *Let $p(x) = x^2 + \alpha x$.*
   Let $f(x) = p(x) \pmod{q}$, and

   $$X = \{f(1), f(2), \ldots, f(m)\}.$$

   Then at least $q/6$ of the intervals $[0, 1)$, $[1, 2)$, $\ldots$, $[q-1, q)$ intersect with X.

2. *Let* $\beta \in \mathsf{R}$. *Let* $p(x) = x^2 + \alpha x + \beta$.
Let $f(x) = p(x) \pmod q$, *and*

$$X = \{f(1), f(2), \ldots, f(m)\}.$$

Then at least $q/6$ *of the intervals* $[0, 1)$, $[1, 2)$, $\ldots$, $[q-1, q)$ *intersect with* X. *(Part 2 is the same proof as Part 1 just a bit messier.)*

Proof: Consider

$$\alpha \bmod q, \quad 2\alpha \bmod q, \quad \ldots, \quad q^2\alpha \bmod q.$$

Map each one to which interval $[0, 1)$, $\ldots$, $[q-1, q)$ that it is in. Some interval has $\geq q$ of these values. Two of those values are $\leq 1/q$ apart. So there exists i, j such that

$$|i\alpha \pmod q - j\alpha \pmod q| \leq \frac{1}{q}.$$

Hence there exists $k \in \mathsf{Z}$ with $|k| \leq q^2$ such that $|k\alpha \pmod q| \leq \frac{1}{q}$. We will assume $k > 0$. the case where $k < 0$ is similar. There are two case depending on if $k \equiv 0 \pmod q$ or not.

Case 1: $k \not\equiv 0 \pmod q$.

We consider $f_1(x) = x^2 \pmod q$. Let

$$X_1 = \{f_1(1), f_1(2), \ldots, f_1(q)\} = \{1^2 \bmod q, 2^2 \bmod q, \ldots, q^2 \bmod q\}$$

By Theorem 2.4, X_1 intersects $\frac{q+1}{2}$ intervals.

We now look at f on multiples of k.

$$Y_1 = \{f_1(k), f_1(2k), \ldots, f_1(qk)\} = \{k^2, (2k)^2, \ldots, (qk)^2\}$$

(Notice that since $k \leq q^2$, $qk \leq q^3 \leq m$. Hence qk is in the range of f that we care about. That is why we need the premise $m \geq q^3$.)

Since $k \not\equiv 0 \pmod q$, $\{k, 2k, \ldots, qk\} = \{1, 2, \ldots, q\}$. Hence $X_1 = Y_1$.

We have shown that

$$\{f_1(k), f_1(2k), \ldots, f_1(qk)\}.$$

hits $(q+1)/2$ intervals. We will show that $\{f(k), f(2k), \ldots, f(qk)\}$ hits $\geq \frac{q}{6}$ intervals.

$f(k) = f_1(k) + k\alpha$. *Key:* Recall $k\alpha \pmod q \leq \frac{1}{q} \leq 1$.

$f(2k) = f_1(2k) + 2k\alpha$. *Key:* Recall $2k\alpha \pmod q \leq \frac{2}{q} \leq 1$.

$\vdots \qquad \vdots$

$f(qk) = f_1(2k) + qk\alpha$. *Key:* Recall $qk\alpha \pmod q \leq \frac{q}{q} \leq 1$.

Recap The set $Y_1 = \{f_1(k), \ldots, f_1(qk)\}$ hits $(q+1)/2$ intervals of length 1.

$Z_1 = \{f(k), \ldots, f(qk)\}$ can be viewed as taking every element in Y_1 and having it jump ≤ 1 to the right or the left. It will either stay in the interval its in originally, or jump to the left interval, or jump to the right interval.

Look at an interval $[a, a+1)$. There is at most one element of Y_1 in $[a-1, a)$, at most one element of Y_1 in $[a, a+1)$, and at most one element of Y_1 in $[a+1, a+2)$. The max number of elements in Z_1 occurs when (a) the element of Y_1 in $[a-1, a)$ jumps into $[a, a+1)$, (b) the element of Y_1 in $[a, a+1)$ stays in $[a, a+1)$, and (c) the element in $[a+1, a+2)$ jumps into $[a, a+1)$, Hence every interval has at most 3 elements of Z_1. Since there are $\frac{q+1}{2}$ elements, at least $\frac{q}{6}$ intervals are hit.

Case 2: $k \equiv 0 \pmod q$.

Recall that $|k\alpha \mod q| \le \frac{1}{q}$ Since $k \equiv 0 \pmod q$, there exists $s \in \mathsf{Z}$ such that $k = sq$. Since $k \le q^2$, $s \le q$.

Hence

$|sq\alpha \bmod q| \le \frac{1}{q}$.

Hence $sq\alpha$ is within $\frac{1}{q}$ of an integer multiple of q. Let $r \in \mathsf{Z}$ and $0 \le \epsilon \le \frac{1}{q}$ be such that

$$sq\alpha = rq + \epsilon$$

$$\alpha = \frac{r}{s} + \frac{\epsilon}{q}$$

Let $\epsilon' = \frac{\epsilon}{q}$ so

$$\alpha = \frac{r}{s} + \epsilon' \text{ where } \epsilon' < \tfrac{1}{q^2}.$$

We can assume r, s have no common factors.

We consider $f_2(x) = x^2 + \frac{r}{s}x \bmod q$.

Let

$$Y_2 = \{f_2(s), f_2(2s), \ldots, f_2((q-1)s)\}$$

(Note that $(q-1)s \le q^2$.)

By Theorem 2.5 Y_2 hits at least $q/2$ intervals. We show that

$|Z_2| = |\{f(s), f(2s), \ldots, f((q-1)s)\}| \le \frac{q}{6}$.

$f(s) = s^2 + \alpha s = s^2 + \frac{r}{s}s + \epsilon' s = f_2(s) + \epsilon' s$. *Key:* $\epsilon' s \le \frac{s}{q^2}$.

$f(2s) = (2s)^2 + 2\alpha k = (2s)^2 + 2\frac{r}{s}s + 2\epsilon' s = f_2(2s) + 2\epsilon' s$. *Key:* $2\epsilon' s \le \frac{2s}{q^2}$.

$\vdots \qquad \vdots$

$f((q-1)s) = ((q-1)s)^2 + s\alpha k = ((q-1)s)^2 + s\frac{r}{s}k + q\epsilon' s = f_2(sk) + q\epsilon' s$. *Key:* $(q-1)\epsilon' s \le \frac{(q-1)s}{q^2} = \frac{s}{q} < 1$.

BILL- WHY IS $\frac{s}{q} < 1$.

By the above Key's, for all i, $|f(is) - f_2(is)| \le 1$.

Recap The set $Y_2 = \{f_2(s), f_2(2s), \ldots, f_2(qs)\}$ hits $q/2$ intervals of length 1.

$Z_2 = \{f(s), f(2s), \ldots, f(qk)\}$ can be viewed as taking every element in Y_2 and adding or subtracting ≤ 1 to it. It is easy to show that Z_2 hits $\ge q/6$ intervals.

■

3 Theorems About Sign Patterns

Notation 3.1

1. If $a \in \mathsf{R}$ then

$$\text{sign}(a) = \begin{cases} - & \text{if } a < 0 \\ 0 & \text{if } a = 0 \\ + & \text{if } a > 0 \end{cases} \tag{1}$$

2. If $\eta \in \{-, 0, +\}^*$ then $\eta(i)$ is the ith character in η.

We will soon define sign-changes but first do an example.

Example 3.2 Let
$p_1(x, y) = x + 2y - 3$
$p_2(x, y) = -2x + 3y - 7$
$p_3(x, y) = 4x - y$

We care about $(\text{sign}(p_1(x, y)), \text{sign}(p_2(x, y)), \text{sign}(p_3(x, y)))$. We look at this sequence for some values of (x, y).

(x, y)	$(p_1(x, y), p_2(x, y), p_3(x, y))$	sign pattern
$(5, 0)$	$(2, -17, 20)$	$(+, +, -)$
$(1, 4)$	$(6, 3, 0)$	$(+, +, 0)$
$(5, 6)$	$(14, 1, 14)$	$(+, +, +)$
$(-5, 0)$	$(-8, 3, -20)$	$(-, +, -)$
$(-3, 3)$	$(0, 8, -15)$	$(0, +, -)$
$(0, \frac{7}{3})$	$(\frac{5}{3}, 0, -\frac{7}{3})$	$(+, 0, -)$
$(0, 2)$	$(1, -1, -2)$	$(+, -, -)$
$(\frac{1}{2}, 2)$	$(\frac{3}{2}, -2, 0)$	$(+, -, 0)$
$(4, 5)$	$(11, 0, 11)$	$(+, 0, +)$
$(5, 5)$	$(12, -2, 15)$	$(+, -, +)$
$(-5, -1)$	$(-10, 0, -19)$	$(-, 0, -)$
$(-5, -5)$	$(-18, -12, -15)$	$(-, -, -)$
$(0, \frac{3}{2})$	$(0, -\frac{5}{2}, -\frac{3}{2})$	$(0, -, -)$
$(0, 0)$	$(-3, -7, 0)$	$(-, -, 0)$
$(5, -5)$	$(-8, -32, 25)$	$(-, -, +)$
$(5, -1)$	$(0, -20, 21)$	$(0, -, +)$
$(-\frac{5}{7}, \frac{13}{7})$	$(0, 0, -\frac{33}{7})$	$(0, 0, -)$
$(\frac{7}{10}, \frac{14}{5})$	$(\frac{33}{10}, 0, 0)$	$(+, 0, 0)$
$(\frac{1}{3}, \frac{4}{3})$	$(0, -\frac{11}{3}, 0)$	$(0, -, 0)$

There are potentially $3^3 = 27$ sign patterns. (p_1, p_2, p_3) has at least 19. We show that there are exactly 19 and then prove a generalization.

Definition 3.3 Let $p_1, \ldots, p_M \in \mathsf{R}[x_1, \ldots, x_N]$. Let

$$X = (p_1, \ldots, p_M).$$

$\eta \in \{-, 0, +\}^M$ is a *sign pattern for* X if there exists $a_1, \ldots, a_N \in \mathsf{R}$ such that for all $1 \le i \le M$

$$\mathrm{sign}(p_i(a_1, \ldots, a_N)) = \eta(i).$$

Theorem 3.4 *Let $p_1(x,y), p_2(x,y), p_3(x,y)$ be 3 linear expressions with 2 variables. Then, there are at most 19 sign patterns.*

Proof: For $i \in \{1,2,3\}$ draw the line $p_i(x,y) = 0$. Then, the 3 expressions will form 3 lines. For any point on the plane, the location of the point with respect to the line determines the sign. If a point is on the line, the sign is 0. If it is on one side of the line, the sign is either positive or negative, and if the point is on the other side of the line, the sign will be flipped. Therefore, each region divided by the lines will represent a sign pattern with no 0, each line segment will represent a sign pattern with one 0, and the point where lines intersect will represent a sign pattern with at least 2 0s.

We want to obtain the maximum number of regions, line segments, and intersections. Now, start with a single line. There are 2 regions, 1 line segment, and 0 intersections. To obtain the maximum number of components, the second line should intersect the existing line, dividing 2 regions and 1 line segment. It also creates 2 new segments by dividing its own and an intersection. In an optimal situation, with 2 lines there are 4 regions, 4 line segments, and 1 intersection. Finally, by adding another line that intersects the 2 existing lines, it divides 3 regions and 2 line segments, and creates 3 line segments and 2 intersections. There are 7 regions, 9 line segments, and 3 intersections that can be created with 3 lines. In total, there are 19 components, each indicating a sign pattern. ∎

DANESH WILL REWRITE 3.4 and 3.5

We will generalize Theorem 3.4 to n linear functions. We will then further generalize to n polynomials.

We need a lemma about lines in the plane before we can obtain a lemma about sign changes.

Lemma 3.5 *Let $n \ge 1$.*

1. *There is a way to place n lines in the plane so that there are $\frac{n^2+n+2}{2}$ regions, n^2 line segments, and $\frac{n^2-n}{2}$ intersections, which are in total $2n^2+1$ components.*
2. *For all sets of n lines in the plane there are $\le \frac{n^2+n+2}{2}$ regions, $\le n^2$ line segments, and $\le \frac{n^2-n}{2}$ intersections, which are in total $\le 2n^2+1$ components.*

Proof: We prove part 2. Part 1 is similar. We prove this by induction on n.

Base case: n**=1** With one line, there are $\frac{1^2+1+2}{2} = 2$ regions, $1^2 = 1$ line segment, and $1^2 - 1 = 0$ intersection.

Induction Hypothesis (IH) For any set of n lines in the plane there are $\le \frac{n^2+n+2}{2}$ regions, $\le n^2$ line segments, and $\le \frac{n^2-n}{2}$ intersections.

Induction Step Assume there is a set of $n+1$ lines in the plane. View these are n lines plus another line L. By the IH the n lines form $\le \frac{n^2+n+2}{2}$ regions, $\le n^2$ line segments, and $\le \frac{n^2-n}{2}$ intersections.

We look at how many new regions, line segments, and intersections can be created.

1. We show that at most $n+1$ new regions are formed, so the total number of regions is at most $\frac{n^2+n+2}{2} + n + 1 = \frac{(n+1)^2+(n+1)+2}{2}$,

   Let L_1 be the first line that L hits. The L may have already divided an region in two. For every line that is encountered by L a new region is created. The case that maximizes the number of regions is when L hits all of the lines. That creates n regions. Upon leaving the last line it may create another region.

2. By a proof similar to the one for regions, one can show that at most $n+1$ new line segments are formed, so the total number of line segments is at most $n^2+n+n+1 = (n+1)^2$.

3. Line L intersects at most n lines, so there are most n new intersections. Hence the number of intersections is at most $\frac{n^2-n}{2} + n = \frac{(n+1)^2-(n+1)}{2}$.

■

Theorem 3.6 *Let $p_1(x,y), \ldots, p_n(x,y) \in \mathsf{R}[x,y]$ be linear. Then there are at most $2n^2+1$ sign patterns.*

Proof: For $i \in \{1, \ldots, n\}$ draw the line $p_i(x,y) = 0$. Then, the n expressions will form n lines. As in the proof of Theorem 3.4 the number of sign patterns is bounded above by the sum of the number of regions, line segments, and intersections. By Lemma 3.5 this sum is bounded by $2n^2+1$. ■

We will present a known generalization of Theorem 3.6.

Let $p_1, \ldots, p_M \in \mathsf{R}[x_1, \ldots, x_N]$. An obvious bound on the number of sign patterns is 3^M The following lemma, due to Oleinik-Petrovsky-Thom-Milnor (see the the book by Basu-Pollack-Roy [1]), shows that, if $N \ll M$, there are far less than 3^M sign patterns. We omit the proof.

Lemma 3.7 *Let $D, M, N \in \mathsf{N}$. Let $p_1, \ldots, p_M \in \mathsf{R}[x_1, \ldots, x_N]$. Assume that all of the p_i's are of degree $\le D$. The number of sign patterns for $(p_1, \ldots, p_N)$ is at most $\left(\frac{50DM}{N}\right)^N$.*

4 Our Goal

Notation 4.1 Let $m \in \mathsf{N}$. Later we will take m large and a cube of a primes.

1. Let E be the set with one linear equation: $y_1 + y_3 = 2y_2 + 2$.

2. Let F be a set of polynomials $\{p_i(z_1, z_2)\}_{i=1}^{m}$. We will motivate and specify F later.

3. Let G be the set with $m-2$ linear equations: $(\forall 2 \le i \le m)[y_{i-1} + y_{i+1} = 2y_i + 2]$.

We will show that there is a 2-coloring of R such that

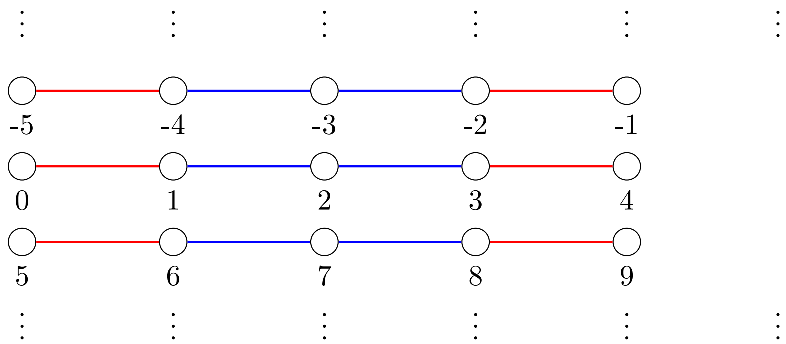

Figure 1: COL from COL$'$

- There is no Red solution to E.
- There is no Blue solution to G. (For this part we will need a result about F.)

Note 4.2 Our proof can surely be modified to apply to some other sets E and G of linear equations. (Such a modification will need a different F.) The next version of this writeup will have that generalization.

5 An Interesting 2-Coloring of R

Definition 5.1 Let r, b be such that $0 \le r, b \le 1$ and $r + b = 1$. We will form two colorings, $\mathrm{COL}'_{r,b}$ and $\mathrm{COL}_{r,b}$ though we will never use the subscripts—they are understood.

$\mathrm{COL}' \colon \mathsf{Z}_q \to [2]$ is defined as follows: For all $x \in \mathsf{Z}_q$:

- $\Pr(\mathrm{COL}'(x) = \mathrm{RED}) = r$.
- $\Pr(\mathrm{COL}'(x) = \mathrm{BLUE}) = b$.

Let $\mathrm{COL} \colon \mathsf{R} \to [2]$ be defined as follows:

$$\mathrm{COL}(z) = \mathrm{COL}'(\lfloor z \rfloor \bmod q).$$

Example 5.2 We take $q = 5$. Let COL$'$ be defined as follows:

$\mathrm{COL}'(0) = \mathrm{RED}$
$\mathrm{COL}'(1) = \mathrm{BLUE}$
$\mathrm{COL}'(2) = \mathrm{BLUE}$
$\mathrm{COL}'(3) = \mathrm{RED}$
$\mathrm{COL}'(4) = \mathrm{RED}$
See Figure 1 for what COL looks like

6 What We Need So That There Is No Red Solution to E

Lemma 6.1 *Let* $0 \le b, r \le 1$, $b + r = 1$. *Let* q *be a prime. Let* $\mathrm{COL}'_{b,r} \colon \mathsf{Z}_q \to [2]$ *and* $\mathrm{COL}_{b,r} \colon \mathsf{R} \to [2]$ *be as defined in Definition 5.1.*

1. *Assume there is a* COL*-RED solution to* $y_1 + y_3 = 2y_2 + 2$. *Then there is a* COL′*-RED solution to either*
    - $n_1 + n_3 = 2n_2 + 1$, *or*
    - $n_1 + n_3 = 2n_2 + 2$, *or*
    - $n_1 + n_3 = 2n_2 + 3$.
2. *The probability that there is a* COL*-RED solution to* $y_1 + y_3 = 2y_2 + 2$ *is bounded above by the probability that there is a* COL′*-RED solution to either*
    - $n_1 + n_3 = 2n_2 + 1$, *or*
    - $n_1 + n_3 = 2n_2 + 2$, *or*
    - $n_1 + n_3 = 2n_2 + 3$.

    This follows from Part 1.

Proof:

Assume there is a COL-RED solution to $y_1+y_3 = 2y_2+2$. Let $n_1, n_2, n_3 \in \mathsf{Z}$ and $0 \le \epsilon_1, \epsilon_2, \epsilon_3 < 1$ be such that

$y_1 = n_1 + \epsilon_1$,
$y_2 = n_2 + \epsilon_2$,
$y_3 = n_3 + \epsilon_3$.

Since $\mathrm{COL}(y_i) = \mathrm{COL}'(n_i)$, $\mathrm{COL}'(n_i) = \mathrm{RED}$.

Note that

$$n_1 + \epsilon_1 + n_3 + \epsilon_3 = 2n_2 + 2\epsilon_2 + 2$$

$$n_1 + n_3 = 2n_2 + (2\epsilon_2 - \epsilon_1 - \epsilon_3 + 2).$$

1. Since $0 \le \epsilon_1, \epsilon_2, \epsilon_3 < 1$, $-2 < 2\epsilon_2 - \epsilon_1 - \epsilon_3 < 2$.
2. Since $n_1, n_2, n_3 \in \mathsf{Z}$, $2\epsilon_2 - \epsilon_1 - \epsilon_3 \in \mathsf{Z}$. Therefore $2\epsilon_2 - \epsilon_1 - \epsilon_3 \in \{-1, 0, 1\}$.
3. $2\epsilon_2 - \epsilon_1 - \epsilon_3 + 2 \in \{1, 2, 3\}$.

Hence n_1, n_2, n_3 is a COL′-RED solution to either

- $n_1 + n_3 = 2n_2 + 1$, or
- $n_1 + n_3 = 2n_2 + 2$, or
- $n_1 + n_3 = 2n_2 + 3$.

$\blacksquare$

Note 6.2 The following is a generalization of Lemma 6.1 that is proven in a similar manner: Let $a \in \mathsf{Z}$. If there is a COL-RED solution to $y_1 + y_3 = 2y_2 + a$ then there is a a COL$'$-RED solution to either $n_1 + n_3 = 2n_2 + a - 1$, or $n_1 + n_3 = 2n_2 + a$, or $n_1 + n_3 = 2n_2 + a + 1$. We will not need this.

Lemma 6.3 *Let* $0 \le b, r \le 1$, $b + r = 1$. *Let* q *be a prime. Let* $\mathrm{COL}'_{b,r} \colon \mathsf{Z}_q \to [2]$ *and* $\mathrm{COL}_{b,r} \colon \mathsf{R} \to [2]$ *be as defined in Definition 5.1. Then the probability that there is a* COL-*RED solution to*

$$y_1 + y_3 = 2y_2 + 2$$

is $\le 3q^2r^3 + 9qr^2$.

Proof:

By Lemma 6.1 the probability that there is a COL-RED solution to $y_1 + y_3 = 2y_2 + 2$. is bounded by the probability that there is a COL$'$-RED solution to

- $n_1 + n_3 = 2n_2 + 1$, or
- $n_1 + n_3 = 2n_2 + 2$, or
- $n_1 + n_3 = 2n_2 + 3$.

All $\equiv$ are mod q.

To get a bound on the probability that there is a COL$'$-RED solution to $n_1 + n_3 = 2n_2 + 1$, we need to upper bound the number of solutions to $n_1 + n_3 = 2n_2 + 1$. There are several types of solutions. We list them and the probability that they occur.

- Solutions where n_1, n_2, n_3 are all different. n_1, n_2 determine n_3. Hence there are $\le q^2$ such solutions. The probability that any of them is RED is $\le q^2r^3$.
- Solutions where $n_1 = n_2$. Then n_1 determines n_3. Hence there are $\le q$ such solutions. The probability that any of them is RED is $\le qr^2$.
- Solutions where $n_1 = n_3$. Then n_1 determines n_2. Hence there are $\le q$ such solutions. The probability that any of them is RED is $\le qr^2$.
- Solutions where $n_2 = n_3$. Then n_2 determines n_1. Hence there are $\le q$ such solutions. The probability that any of them is RED is $\le qr^2$.
- There are no solutions where $n_1 = n_2 = n_3$. So this case does not contribute to the probability.

Hence the probability that there is a COL$'$-RED solution to $n_1 + n_3 = 2n_2 + 1$ is

$$\le q^2r^3 + 3qr^2$$

The same reasoning applies to $n_1 + n_3 = n_2 + 2$ and $n_1 + n_3 = n_2 + 3$. Hence the probability that there is a COL′-RED solution to any of the three equations is

$$\leq 3q^2r^3 + 9qr^2$$

▮

Note 6.4 Using Note 6.2 the following is generalization of Lemma 6.3 can be obtained, in a similar manner of the proof of Lemma 6.3: Let $a \in \mathsf{Z}$. Then the probability that there is a $\mathrm{COL}_{b,r}$-RED solution to $y_1 + y_3 = 2y_2 + a$ is $\leq 3q^2r^3 + 9qr^2$.

7 What We Need So That There Is No Blue Solution to G

7.1 Mono Image Solutions and Mono Solutions

We need to consider the following variant on mono solutions.

Definition 7.1 Let $p_1(z_1, z_2)$, …, $p_m(z_1.z_2) \in \mathsf{Z}[z_1, z_2]$. We denote this set of polynomials by F.

1. An *image-solution of F* is just $(a, d; y_1, \ldots, y_m)$ where $(\forall 1 \leq i \leq m)[y_i = p_i(a, d)]$.
2. Assume the reals are 2-colored. Then a *mono image-solution of F* is an image-solution $(a, d; y_1, \ldots, y_m)$ where $y_1, \ldots, y_m$ are all the same color. Note that a, d need not be that color.
3. A *Red image-solution of F* and a *Blue image-solution of F* have the obvious meaning.

Lemma 7.2 *Let $m \in \mathsf{N}$. We assume m is a cube. Let G be the set of equations*

$$(\forall 2 \leq i \leq m - 1)[y_{i+1} = 2y_i - y_{i-1} + 2].$$

There exists a set of polynomials F:

$$p_1(z_1, z_2), p_2(z_1, z_2), \ldots, p_m(z_1, z_2) \in \mathsf{Z}[z_1, z_2]$$

such that the following holds:

C1) For all i, $p_i(z_1, z_2)$ is linear in z_1, z_2.

C2) The coefficients of $p_i(z_1, z_2)$ are quadratic polynomials in i with coefficients in Z.

C3) If $a, d \in [0, m^{1/3}]$ then, for all i, $0 \leq p_i(a, d) \leq 2m^2$

C4) (a) If $(a, d; y_1, \ldots, y_m)$ is an image-solution of F then $(y_1, \ldots, y_m)$ is a solution to G.

(b) If $(y_1, \ldots, y_m)$ is a solution to G then $(a, d; y_1, \ldots, y_m)$, where $a = y_1$ and $d = y_2 - y_1$, is an image-solution of F.

Proof:

We motivate the definition of $\{p_i(z_1,z_2)\}_{i=1}^m$ by assuming that G has a solution $(y_1,\ldots,y_m)$. Let

$y_1 = a$

$y_2 = a+d$.

Then

$y_3 = 2y_2 - y_1 + 2 = 2(a+d) - a + 2 = a + 2d + 2$

$y_4 = 2y_3 - y_2 + 2 = 2a + 4d + 4 - (a+d) + 2 = a + 3d + 6$

One can verify by algebra that

$(\forall 1 \le i \le m)[y_i = a + (i-1)d + (i^2 - 3i + 2)]$.

We define p_i as follows. For $1 \le i \le m$ let

$$p_i(z_1,z_2) = z_1 + (i-1)z_2 + (i^2 - 3i + 2).$$

Let

$$F = \{p_1(z_1,z_2),\ldots,p_m(z_1,z_2)\}.$$

Properties 1,2,3,4 for F are easy to verify.

■

7.2 What We Need So That There is No Blue Image-Solution to F

Lemma 7.3 *Let q be a prime and let $m \ge q^3$. Let $p_1(z_1,z_2),\ldots,p_m(z_1,z_2) \in \mathsf{Z}[z_1,z_2]$ be such that the following hold:*

C1) For all i, $p_i(z_1,z_2)$ is linear in z_1,z_2.

C2) The coefficients of $p_i(z_1,z_2)$ are quadratic polynomials in i over Z.

C3) If $a,d \in [0,q]$ then, for all i, $0 \le p_i(a,d) \le 2m^2$

Let b,r be such that $0 \le b,r \le 1$ and $b+r=1$. Let COL *be as in Definition 5.1.*

THEN

the probability there is a blue image-solution to $p_1,\ldots,p_m$ is bounded above by $2500m^6b^{m/6}$.

Proof:

The proof is in two parts

PART ONE: The Set of Intervals Mod q.

Fix $a,d \in \mathsf{R}$. We want to bound

$$\Pr(\mathrm{COL}(p_1(a,d)) = \cdots = \mathrm{COL}(p_m(a,d)) = \mathrm{BLUE}).$$

Recall that $\mathrm{COL}(z) = \mathrm{COL}'(z \bmod q)$. Hence, in order to have

$$\mathrm{COL}(p_1(a,d)) = \cdots = \mathrm{COL}(p_m(a,d)) = \mathrm{BLUE})$$

we need to have the following happen:

- $p_1(a,d) \bmod q$ is in an interval that COL$'$ colors BLUE. This occurs with probability b.
- $p_2(a,d) \bmod q$ is in an interval that COL$'$ colors BLUE. This occurs with probability b.
- Etc until $p_m(a,d) \bmod q$ is in an interval that COL$'$ colors BLUE. This occurs with probability b.

At first glance you might think the probability of this happening is b^m which is small, so good news for us. Alas no. For one thing, the events are not independent. More concretely, here is an extreme possibility (that we later show cannot happen): all of the $p_i(a,d) \bmod q$ are in the same interval. This raises the question: how many distinct intervals do we get?

Let $F(x) = p_x(a,d)$. By premise 2, $F(x)$ is quadratic. Also recall that q is prime and $m \ge q^3$. Hence $F(x), m, q$ satisfy the premise of Theorem 2.6. Therefore

$$\{F(1) \bmod q, \dots, F(m) \bmod q\}$$

will intersect $\ge q/6$ intervals. Hence

$$\{p_1(a,d) \bmod q, \dots, p_m(a,d) \bmod q\}$$

will intersect $\ge q/6$ intervals.

Hence

$$\Pr(\mathrm{COL}(p_1(a,d)) = \cdots = \mathrm{COL}(p_m(a,d)) = \mathrm{BLUE}) \le b^{q/6}.$$

PART TWO: How Many Sets of Intervals?

The statement

$$\Pr(\mathrm{COL}(p_1(a,d)) = \cdots = \mathrm{COL}(p_m(a,d)) = \mathrm{BLUE}) \le b^{q/6}.$$

is not about a, d: its about interval patterns. That is, we map a, d to the set of intervals mod q that

$$p_1(a,d) \bmod q, \dots, p_m(a,d) \bmod q$$

are in and then we look at the probability that all of those intervals are the same color.

So the question is: how many sets of intervals can there be?

Since the coloring is mod q we can assume $a, d \in [0, q)$. By premise 3, $0 \le p_i(a,d) \le 2m^2$. We now ask about the intervals (not mod q).

Consider the following questions:

- Of the intervals $[0,1)$, $[1,2)$, $\dots$, $[2m^2-1, 2m^2)$ which one has $p_1(a,d)$? There are $2m^2$ possibilities. Note that which interval can be determined from the sign changes of the following sequence:

  $$p_1(a,d) - 1, p_1(a,d) - 2, \dots, p_1(a,d) - 2m^2.$$

- Of the intervals $[0,1)$, $[1,2)$, $\dots$, $[2m^2-1, 2m^2)$ which one has $p_2(a,d)$? There are $2m^2$ possibilities.

  Note that which interval can be determined from the sign changes of the following sequence:

  $$p_2(a,d) - 1, p_2(a,d) - 2, \dots, p_2(a,d) - 2m^2.$$

- Etc until Of the intervals $[0,1)$, $[1,2)$, …, $[2m^2-1,2m^2)$ which one has $p_m(a,d)$? There are $2m^2$ possibilities.

  Note that which interval can be determined from the sign changes of the following sequence:

$$p_m(a,d)-1, p_m(a,d)-2, \ldots, p_m(a,d)-2m^2.$$

At a first glance it would seem like there are $(2m^2)^m$ possibilities. There are far less. Consider the sequence of polynomials

$$p_1(z_1,z_2)-1, \ldots, p_1(z_1,z_2)-2m^2,$$

$$p_2(z_1,z_2)-1, \ldots, p_2(z_1,z_2)-2m^2,$$

$$\vdots$$

$$p_m(z_1,z_2)-1, \ldots, p_m(z_1,z_2)-2m^2.$$

BILL- BE MORE CLEAR WHICH SEQUENCE WE ARE DOING SIGN CHANGES ON.

The maximum number of sign changes in this sequence is an upper bound on the number of ways $p_1(a,d)$, …, $p_m(a,d)$ can be in the intervals.

By Lemma 3.7 the number of sign changes is $\le 625(2m^3)^2 = 2500m^6$. Hence the number of ways $p_1(a,d), \ldots, p_m(a,d)$ are in the intervals is $\le 2500m^6$.

COMBINE PARTS ONE AND TWO

Map $(a,d) \in \mathsf{R}^2$ to the set of $\ge q/6$ intervals that contain

$$p_1(a,d) \bmod q, \ldots, p_m(a,d) \bmod q.$$

There are at most $2500m^6$ sets of intervals. Each one has probability $\le b^{q/6}$ of having them all be blue. By the union bound the probability that one of those sets has every interval blue is

$$\le 2500m^6 b^{q/6}.$$

∎

7.3 What We Need so There is no Blue Solution to G

By combining Lemmas 7.2 and 7.3 we obtain the following.

Lemma 7.4 *Let $m \in \mathsf{N}$. We assume m is a cube. Let G be the set of equations*

$$(\forall 2 \le i \le m-1)[y_{i+1} = 2y_i - y_{i-1} + 2].$$

Let b, r be such that $0 \le b, r \le 1$ and $b + r = 1$. Let COL be as in Definition 5.1.

THEN

the probability there is a blue solution to G is bounded above by $2500m^6 b^{m/6}$.

8 No Red Sol to E and No Blue Sol to G

Theorem 8.1 *Let q be a large prime to be determined later. Let $m = q^3$.*

Let E be the set of equations (actually only one equation) $y_1 + y_3 = 2y_2 + 2$.

Let G be the set of equations $(\forall 2 \le i \le m-1)[y_{i-1} + y_{i+1} = 2y_i + 2]$.

Then there exists a 2-coloring of R such that the following hold:

1. *There is no RED solution to E.*
2. *There is no BLUE solution to G.*

Proof: Let $0 \le b, r \le 1$ such that $b + r = 1$. We will pick b, r later. Let q a prime such that $m \ge q^3$. Let COL be as in Definition 5.1 with parameters r, s, q.

By Lemma 6.3,

$$\Pr(\text{there is a RED solution to } y_1 + y_3 = 2y_2 + 2) \le 3q^2r^3 + 9qr^2.$$

By Lemma 7.4,

$$\Pr(\text{there is a BLUE solution to } \{p_i(x,y)\}_{i=1}^{m}) \le 2500m^6b^{q/6}.$$

Hence we want to pick b, r such that

$$3q^2r^3 + 9qr^2 + 2500m^6b^{q/6} < 1$$

Choose $r = q^{-3/4}$ and $b = 1 - r$

Then,

$$3q^2r^3 + 9qr^2 = 3q^{-1/4} + 9q^{-1/2} < 12q^{-1/4} < \frac{1}{2}$$

for a sufficiently large q.

Next, note that, using $b = 1 - q^{-3/4}$ and $m = q^3$ that

$$2500m^6b^{q/6} = 2500m^6(1 - q^{-3/4})^{q/6} = 2500q^{18}(1 - q^{-3/4})^{q/6}$$

For sufficiently large q, we have $(1 - q^{-3/4}) \sim e^{-q^{-3/4}}$, hence $(1 - q^{-3/4})^{q/6} = e^{-q^{1/4}/6}$. We use this to obtain

$$2500q^{18}(1 - q^{-3/4})^{q/6} \sim 2500q^{18}e^{-q^{1/4}/6}$$

Therefore, if q is sufficiently large, then taking $r = q^{-3/4}$ and $b = 1 - r$ make the probability of BLUE solution to G is $< \frac{1}{2}$.

We already established that if q is sufficiently large, then taking $r = q^{-3/4}$ and $b = 1 - r$ make the probability of RED solution to E is $< \frac{1}{2}$.

Hence, for large enough q, the probability of having a RED solution to E or a BLUE solution to G less than 1.

This shows that there exists a 2-coloring of R such that:

There is no Red solution to E and there is no Blue solution to G. ∎

9 The ℓ_3-ℓ_m Theorem

Recall that in an earlier chapter we showed the following:

Theorem 9.1 *There exists constants $c_1 < c_2$ such that the following are true:*

1. *There is a 2-coloring of R^n with no RED ℓ_2 and no BLUE $\ell_{2^{c_1 n}}$.*
2. *For every 2-coloring of R^n there is either a RED ℓ_2 or a BLUE $\ell_{2^{c_2 n}}$.*

Note that the length of the BLUE line depends on n.
What about ℓ_3 and ℓ_m? Perhaps surprisingly, the next result does not depend on n.

Theorem 9.2 *Let $n \geq 2$. There exists* $\mathrm{COL}\colon \mathsf{R}^n \to [2]$ *such that*

1. *there is no* RED ℓ_3*, and*
2. *there is no* BLUE ℓ_m *where m will be determined later. m will be around* 10^{50}.

Proof:

Implications of ℓ_3. No Coloring Involved
Let

- $\vec{0}$ be $(0,\ldots,0)$. (There are n 0's.)
- $\vec{a}_1, \vec{a}_2, \vec{a}_3$ be an ℓ_3. ($\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ are in R^n).
- $x_1 = d(\vec{0}, \vec{a}_1)$, $x_2 = d(\vec{0}, \vec{a}_2)$, $x_3 = d(\vec{0}, \vec{a}_3)$.

We know
$d(\vec{a}_1, \vec{a}_2) = d(\vec{a}_2, \vec{a}_3) = 1$,
Figure 2 summarizes what is going on.

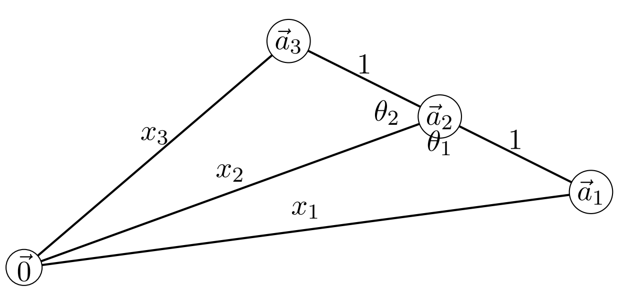

Figure 2: $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ form an ℓ_3

If we apply the law of cosines to the bottom triangle we get:

$$(1) \qquad x_1^2 = x_2^2 + 1 - 2x_2 \cos(\theta_1).$$

If we apply the law of cosines to the top triangle we get:

$$(2) \qquad x_3^2 = x_2^2 + 1 - 2x_2\cos(\theta_2).$$

Let $\theta_2 = \pi - \theta_2$. Hence $\cos(\theta_2) = -\cos(\theta_1)$. So

$$(3) \qquad x_3^2 = x_2^2 + 1 + 2x_2\cos(\theta_1).$$

Add the equations (1) and (3) to get

$$x_1^2 + x_3^2 = 2x_2^2 + 2.$$

End of Implications of ℓ_3

Our First Plan:

1) Find $\mathrm{COL}' \colon \mathsf{R} \to [2]$ such that there is no RED solution to

$$x_1^2 + x_3^2 = 2x_2^2 + 2.$$

2) Define $\mathrm{COL} \colon \mathsf{R}^2 \to [2]$ by $\mathrm{COL}(\vec{a}) = \mathrm{COL}'(d(\vec{0}, \vec{a}))$.

It is easy to show that if COL has a RED ℓ_3 then COL$'$ has RED sol to $x_1^2 + x_3^2 = 2x_2^2 + 2$. Hence COL does not have a RED ℓ_3.

This plan is promising but there is an even easier one. The fact that x_1, x_2, x_3 are squared is not important.

Our Second Plan:

1) Find $\mathrm{COL}' \colon \mathsf{R} \to [2]$ such that there is no RED solution to

$$y_1 + y_3 = 2y_2 + 2.$$

2) Define $\mathrm{COL} \colon \mathsf{R}^2 \to [2]$ by $\mathrm{COL}(\vec{a}) = \mathrm{COL}'(d(\vec{0}, \vec{a})^2)$.

It is easy to show that if COL has a RED ℓ_3 then COL$'$ has a RED sol to $x_1^2 + x_3^2 = 2x_2^2 + 2$ Hence COL does not have a RED ℓ_3.

The Second Plan is the plan, except that we also need COL to not have a BLUE ℓ_m

Implications of ℓ_m. No Coloring Involved

Let

- $\vec{0}$ be $(0, 0)$.
- $\vec{a}_1, \ldots, \vec{a}_m$ be an ℓ_m.
- $x_1 = d(\vec{0}, \vec{a}_1)$, ..., $x_m = d(\vec{0}, \vec{a}_m)$.

We know $d(\vec{a}_1, \vec{a}_2) = \cdots = d(\vec{a}_{m-1}, \vec{a}_m) = 1$.

By using the reasoning about ℓ_3, applied to all consecutive triples of $\vec{a}_1, \ldots, \vec{a}_m$, we get that we need that COL$'$ has no BLUE solution to

$$(\forall 2 \le i \le m-1)[y_{i-1} + y_{i+1} = 2y_i + 2].$$

End of implications of ℓ_m.

Final Plan

By Theorem 8.1 there is a 2-coloring of R such that

- There is no RED solution to $y_1 + y_3 = 2y_2 + 2$.
- There is no BLUE solution to $(\forall 2 \le i \le m-1)[y_{i-1} + y_{i+1} = 2y_i + 2]$.

That provides the desired coloring of R^2.

∎

References

[1] S. Basu, R. Pollack, and M.-F. Roy. *Algorithms in real algebraic geometry.* Springer, 2006.

[2] D. Conlon and Y. H. Wu. More on lines in Euclidean Ramsey theory. *C.R. Math. Acad. Paris*, 361:897–901, 2023.