# The Infinite $a$-ary Can Ramsey Thm

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# Min-Homog, Max-Homog, Rainbow

**Def:** Let $\mathrm{COL} : \binom{\mathbb{N}}{2} \to \omega$. Let $V \subseteq \mathbb{N}$. Assume $a < b$ and $c < d$.

- $V$ is *homog* if $\mathrm{COL}(a, b) = \mathrm{COL}(c, d)$ iff *TRUE*.
- $V$ is *min-homog* if $\mathrm{COL}(a, b) = \mathrm{COL}(c, d)$ iff $a = c$.
- $V$ is *max-homog* if $\mathrm{COL}(a, b) = \mathrm{COL}(c, d)$ iff $b = d$.
- $V$ is *rainb* if $\mathrm{COL}(a, b) = \mathrm{COL}(c, d)$ iff $a = c$ and $b = d$.

**Can Ramsey Thm for $\binom{\mathbb{N}}{2}$:** For all $\mathrm{COL} : \binom{\mathbb{N}}{2} \to \omega$, there exists an infinite set $V$ such that $V$ is homog OR min-homog OR max-homog OR rainb.

# Restate So We Can Generalize

**Def:** Let $\mathrm{COL} : \binom{\mathbb{N}}{2} \to \omega$. Let $V \subseteq \mathbb{N}$. Assume $a_1 < a_2$ and $b_1 < b_2$.

- $V$ is *homog* if $\mathrm{COL}(a_1, a_2) = \mathrm{COL}(b_1, b_2)$ iff *TRUE*. So $\mathrm{COL}(x, y)$ does not depend on the first or second coordinate. We call this $\emptyset$-homog.
- $V$ is *min-homog* if $\mathrm{COL}(a_1, a_2) = \mathrm{COL}(b_1, b_2)$ iff $a_1 = b_1$. So $\mathrm{COL}(x, y)$ depend on the first coordinate only. We call this $\{1\}$-homog.
- $V$ is *max-homog* if $\mathrm{COL}(a_1, a_2) = \mathrm{COL}(b_1, b_2)$ iff $a_2 = b_2$. So $\mathrm{COL}(x, y)$ depend on the second coordinate only. Can call this $\{2\}$-homog.
- $V$ is *rainb* if $\mathrm{COL}(a_1, a_2) = \mathrm{COL}(b_1, b_2)$ iff $a_1 = b_1$ and $a_2 = b_2$. So $\mathrm{COL}(x, y)$ depend on the first and second coordinate only. Can call this $\{1, 2\}$-homog.

**Can Ramsey Thm for $\binom{\mathbb{N}}{2}$:** For all $\mathrm{COL} : \binom{\mathbb{N}}{2} \to \omega$, there exists $A \subseteq \{1, 2\}$ and an infinite set $V$ such that $V$ is $A$-homog.

# All 8 Cases For 3-Ary Can Ramsey

$\mathrm{COL} : \binom{\mathbb{N}}{3} \rightarrow \omega$. $V \subseteq \mathbb{N}$. $a_1 < a_2 < a_3$ and $b_1 < b_2 < b_3$.

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Define $\mathrm{COL} : \binom{\mathbb{N}}{3} \to \omega$ by

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The rest of the cases are similar.

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3. One is similar to the proof of 3-hypergraph Ramsey.

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We might later discuss why he did this.