Constructive Ramsey Lower Bounds AND Chromatic Number of $\mathbb{R}^n$!

Exposition by
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and
Neal Shandilya-Howard HS

Credit Where Credit is Due

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3) A great set of slides by Kelin Zhu.
(An undergraduate working with William Gasarch.)

Easy But Weak Lower Bounds on $R(k)$.

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Lemma Needed For $R(k) = \Omega(k^{\log_4 17})$

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Exercise Show that if $R(k_0) > m_0$ then $R(k) = \Omega(k^{\log_{k_0} m_0})$.

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Exercise Use the last exercise to obtain a better lower bound than $R(k) = \Omega(k^{\log_4 17})$.

We Will Prove Two Combinatorial Lemmas

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If I have $w_1, \ldots, w_m$ then these are all distinct.

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Next slide is examples.

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Rest of proof on next slide. (So write down theorem.)

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Next slide is examples.

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Case 4 $1 \le \lambda < v$. Next slide.

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End of Proof

Constructive Lower Bounds on $R(k)$

We Will Show

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Case 2 on next slide.

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Exercise Show that our proof that $R(k) > \binom{k-1}{3}$ is constructive.

3) There is a nonconstructive proof that shows $R(k) = \Omega(k2^{k/2})$.

We Will Show

$$\chi(\mathbb{R}^n) = \Omega(n^2)$$

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What is $\chi(\mathbb{R}^n)$? The next slide gives some information.

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4) The best known results asymptotically are

$$(1.239 + o(1))^n \le \chi(\mathbb{R}^n) \le (3 + o(1))^n.$$

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Notation We call that last theorem **Oddtown'**.

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We will use $c < \frac{(n-1)(n-2)}{6}$.

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Proof continued on next slide.

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S satisfies premise of Oddtown', so $\exists x, y \in S$, $|x \cap y| = 1$.

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Note For $p = 2$ and $L = \{0\}$, the generic bound is $m \le n + 1$.
However, the polynomials have constant term $\prod_{\ell \in L}(-\ell) = 0$, so the sharper dimension count gives $m \le n$, recovering Oddtown.
Note that, for all p, if $0 \in L$ then $m \le (\sum_{a=0}^{|L|} \binom{n}{a}) - 1$.

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Proof continued on next slide.

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Proof continued on next slide.

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End of Proof

We use Gen of ODD and FISH to get better Lower Bound on $R(k)$.

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Lemma Let $n, p \in \mathbb{N}$ such that $n \geq 4p$. Then $\sum_{a=0}^{p-1} \binom{n}{a} < 2\binom{n}{p-1}$.
Proof Use $1 \leq a \leq p-1$ on the first inequality, and $n \geq 4p$ on the second inequality to get:

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$R(k)$ Grows Faster than Any Polynomial

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Exercise For every fixed $d \in \mathbb{N}$, prove that $R(k)/k^d \to \infty$. Hint: choose a fixed prime $p > d$ and use an argument similar to ours.

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We use Gen of ODD to get better Lower Bound on $\chi(\mathbb{R}^n)$

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Notation We call the above Theorem Gen-Oddtown'.

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Proof continues on next slide.

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Recap

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Note Only used Gen. Oddtown.

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Thm Let $n \in \mathbb{N}$ and p be a prime such that $n \ge 4p$. Let $c < \binom{n}{2p-1} / \sum_{a=0}^{p-1} \binom{n}{a}$. $\forall \mathrm{COL}\colon \mathbb{R}^n \to [c]$ $\exists$ two points, same color, an inch apart.
By Lemma and $n \ge 4p$ we have $\sum_{a=0}^{p-1} \binom{n}{a} \le 2\binom{n}{p-1}$. Hence
$\chi(\mathbb{R}^n) \ge \frac{1}{2} \frac{\binom{n}{2p-1}}{\binom{n}{p-1}}$.

$$\chi(\mathbb{R}^n) \ge \frac{(p-1)!}{2^{p+1}(2p-1)!} n^p.$$

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Better is known. See next slide.

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Question From Bill Frankl-Wilson uses prime powers, not primes. Does this improve the results?