Frankl–Wilson and Constructive Ramsey Lower Bounds

Kelin Zhu

University of Maryland

Nagy's theorem

Theorem (Nagy)

$$R(n+1, n+1) > \binom{n}{3}.$$

This was one of the earlier instances of a *lower bound* on Ramsey numbers.
The construction is a 2-coloring of a graph whose vertex set is

$$V(G) = \binom{[n]}{3}.$$

Two classical theorems

Oddtown

If $S_1, \ldots, S_k \subseteq [n]$ all have odd size and every pairwise intersection has even size, then

$$k \leq n.$$

Fisher's Inequality

If $S_1, \ldots, S_k \subseteq [n]$ all have size a and every pairwise intersection has size b, where $a \neq b$, then

$$k \leq n.$$

Nagy's construction

Construct a graph G with vertex set

$$V(G) = \binom{[n]}{3}.$$

For distinct vertices $u, v \in V(G)$, color the edge uv red if

$$|u \cap v| \text{ is even},$$

and blue otherwise.

Because u and v are distinct,

$$|u \cap v| \in \{0, 1, 2\}.$$

Why there is no monochromatic K_{n+1}

Red clique

A red K_{n+1} would give $n+1$ sets of size 3, each of odd size, with every pairwise intersection even. This contradicts Oddtown.

Blue clique

A blue K_{n+1} would give $n+1$ sets of size 3 with every pairwise intersection equal to 1. This contradicts Fisher's inequality.

Therefore G has no monochromatic K_{n+1}, and hence

$$R(n+1, n+1) > |V(G)| = \binom{n}{3}.$$

Linear algebra theorem

Oddtown and Fisher are subsumed by a linear algebraic theorem.
Replace each set by its incidence vector $v_i \in \mathbf{F}^n$.

Theorem

If $v_1, \ldots, v_k \in \mathbf{F}^n$ satisfy

$$v_i \cdot v_i = a, \qquad v_i \cdot v_j = b \quad (i \neq j),$$

with $a \neq b$ and $a + (k-1)b \neq 0$, then $v_1, \ldots, v_k$ are linearly independent.

- Start with a dependence $c_1 v_1 + \cdots + c_k v_k = 0$ and let $S = c_1 + \cdots + c_k$.
- Dot with each v_i to get

$$(a-b)c_i + bS = 0.$$

- Since $a \neq b$, all c_i are equal; say $c_i = c$.
- Then $(a + (k-1)b)c = 0$, so $c = 0$.

Application to set intersection

Set Intersection Formulation

If p is prime and distinct sets $S_1, \ldots, S_k \subseteq [n]$ satisfy

$$|S_i| \equiv a \pmod{p}, \qquad |S_i \cap S_j| \equiv b \pmod{p} \quad (i \neq j),$$

with $a \neq b$ and $a + (k-1)b \not\equiv 0 \pmod{p}$, then

$$k \leq n.$$

Oddtown is $p = 2$, $a = 1$, $b = 0$; Fisher follows by choosing p very large.

Corollary

Even without the condition $a + (k-1)b \neq 0$, the only possible linear dependence between the v's is

$$v_1 + \cdots + v_k = 0.$$

Hence $k \leq n + 1$, and the same $n + 1$ bound holds for the set version as well.

Modified Frankl–Wilson theorem

Nagy construction doesn't work for $\binom{[n]}{4}$ since there are too many intersection sizes. We now want a tool that deals with many intersection sizes at once.

Theorem (Modified Frankl–Wilson)

Let p be prime and let $B \subseteq \mathbf{Z}/p\mathbf{Z}$ with $|B| = m < p$. Suppose $S_1, \ldots, S_k \subseteq [n]$ are distinct and satisfy

$$|S_i| \notin B \pmod{p} \qquad \text{for all } i,$$

and

$$|S_i \cap S_j| \in B \pmod{p} \qquad \text{for all } i \neq j.$$

Then

$$k \leq 1 + n + \cdots + n^m.$$

The case $m = 1$ is a stronger version of the previous linear algebra theorem. Frankl-Wilson showed a stronger bound $k \leq \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{m}$, but our version can prove similar results in the Ramsey context.

Proof

Work over $\mathbf{F}_p^n$, and let y_i be the incidence vector of S_i. Define

$$F(x, y) = \prod_{b \in B} (x \cdot y - b), \qquad f_i(x) = F(x, y_i).$$

- Since $|S_i| \notin B \pmod{p}$, we have

$$f_i(y_i) \neq 0.$$

- Since $|S_i \cap S_j| \in B \pmod{p}$ for $i \neq j$, we have

$$f_i(y_j) = 0 \qquad (i \neq j).$$

- So the matrix $(f_i(y_j))$ is diagonal with nonzero diagonal, hence

$$f_1, \ldots, f_k \text{ are linearly independent.}$$

Proof

Each f_i is a product of m linear factors, so

$$\deg(f_i) \leq m.$$

There are at most n^r monomials of degree r in n variables. Therefore the space of polynomials of degree at most m has dimension at most

$$1 + n + \cdots + n^m.$$

Conclusion

The independent polynomials $f_1, \ldots, f_k$ live in a vector space of dimension at most $1 + n + \cdots + n^m$, so

$$k \leq 1 + n + \cdots + n^m.$$

Ramsey application

Theorem (Super-polynomial lower bound)

For every positive integer t and all sufficiently large N,

$$R(N, N) > N^t.$$

This is weaker than the probabilistic lower bound, but it gives an explicit coloring.

Fix t, and let q be the smallest prime with $q \geq t$. We will build a 2-coloring on a graph whose vertices are $\binom{[n]}{q^2-1}$.

Ramsey construction from Frankl–Wilson

Let n be chosen later, and define a graph G with vertex set

$$V(G) = \binom{[n]}{q^2 - 1}.$$

For distinct vertices $u, v \in V(G)$, color the edge uv red if

$$|u \cap v| \equiv -1 \pmod{q},$$

and blue otherwise.

Every vertex has size

$$q^2 - 1 \equiv -1 \pmod{q}.$$

Ramsey construction from Frankl–Wilson

Blue clique

Take

$$p = q, \qquad B = \{0, 1, \ldots, q - 2\}.$$

Then a blue clique gives sets of size $q^2 - 1 \notin B \pmod{q}$ whose pairwise intersections lie in B. So Frankl–Wilson gives

$$k \leq 1 + n + \cdots + n^{q-1} < 2n^{q-1}.$$

Red clique

Take a prime $P > q^2 - 1$ and let

$$B = \{q - 1, 2q - 1, \ldots, (q - 1)q - 1\}.$$

A red clique has pairwise intersections in exactly those residues, while each set has size $q^2 - 1 \notin B$. Again,

$$k \leq 1 + n + \cdots + n^{q-1} < 2n^{q-1}.$$

Proof of lower bound

Now define

$$n = \left(\frac{N}{2}\right)^{1/(q-1)}$$

Then $2n^{q-1} = N$.
So every red clique and every blue clique has size $< N$. Hence G has no monochromatic K_N, and therefore

$$R(N, N) > |V(G)| = \binom{n}{q^2 - 1}.$$

Proof of lower bound

Since q depends only on t, there is a constant $c_q > 0$ such that for all sufficiently large n,

$$|V(G)| = \binom{n}{q^2 - 1} \geq c_q n^{q^2-1}.$$

Since n is asymptotic to $N^{\frac{1}{q-1}}$, for some constant c'_q, we have

$$|V(G)| \geq c'_q N^{(q^2-1)/(q-1)} = c'_q N^{q+1}.$$

Since $q + 1 > t$, for all sufficiently large N,

$$R(N, N) > |V(G)| > N^t.$$

Conclusion

The handout defines an explicit super-polynomial function $f(N)$ with

$$R(N, N) > f(N).$$

No new ideas, just more busy work bounding.

Any questions?