BILL, RECORD LECTURE!!!!

BILL RECORD LECTURE!!!

The Distinct Volumes Problem

David Conlon- Cambridge (Prof)
Jacob Fox-MIT (Prof)
William Gasarch-U of MD (Prof)
David Harris- U of MD (Grad Student)
Douglas Ulrich- U of MD (Ugrad Student)
Sam Zbarsky- Mont. Blair. (High School Student- now CMU)

Darling Wants an Actual Coloring

1. **Infinite Ramsey Thm:** For any 2-coloring of the EDGES of K_ω there exists an infinite *monochromatic* K_ω.

Darling Wants an Actual Coloring

1. **Infinite Ramsey Thm:** For any 2-coloring of the EDGES of K_ω there exists an infinite *monochromatic* K_ω.
2. **Infinite Can Ramsey Thm:** For any ω-coloring of the EDGES of K_ω there exists an infinite H such that either (1) H homog, (2) H min-homog, (3) H max-homog, (4) H rainbow.

Darling Wants an Actual Coloring

1. **Infinite Ramsey Thm:** For any 2-coloring of the EDGES of K_ω there exists an infinite *monochromatic* K_ω.
2. **Infinite Can Ramsey Thm:** For any ω-coloring of the EDGES of K_ω there exists an infinite H such that either (1) H homog, (2) H min-homog, (3) H max-homog, (4) H rainbow.
3. **Darling:** Give me an **example** of an actual coloring.

Darling Wants an Actual Coloring

1. **Infinite Ramsey Thm:** For any 2-coloring of the EDGES of K_ω there exists an infinite *monochromatic* K_ω.
2. **Infinite Can Ramsey Thm:** For any ω-coloring of the EDGES of K_ω there exists an infinite H such that either (1) H homog, (2) H min-homog, (3) H max-homog, (4) H rainbow.
3. **Darling:** Give me an **example** of an actual coloring.

Bill thinks of one— next page.

Points and Distances

Let $p_1, p_2 \ldots .$ be an infinite set of points in $\mathbb{R}$.

Points and Distances

Let $p_1, p_2, \ldots$ be an infinite set of points in $\mathbb{R}$.
Let $\mathrm{COL} : \binom{\mathbb{N}}{2} \to \mathbb{R}$ be

$$\mathrm{COL}(i,j) = |p_i - p_j| \text{ (Distance Between } p_i \text{ and } p_j).$$

Points and Distances

Let $p_1, p_2, \ldots$ be an infinite set of points in $\mathbb{R}$.
Let $\mathrm{COL} : \binom{\mathbb{N}}{2} \rightarrow \mathbb{R}$ be

$$\mathrm{COL}(i,j) = |p_i - p_j| \text{ (Distance Between } p_i \text{ and } p_j).$$

Result: For any infinite set of points in the plane there is an infinite subset where all distances are distinct. (Already known by Erdös via diff proof.)

Points and Distances

Let $p_1, p_2 \ldots.$ be an infinite set of points in $\mathbb{R}$.
Let $\mathrm{COL} : \binom{\mathbb{N}}{2} \rightarrow \mathbb{R}$ be

$$\mathrm{COL}(i,j) = |p_i - p_j| \text{ (Distance Between } p_i \text{ and } p_j).$$

Result: For any infinite set of points in the plane there is an infinite subset where all distances are distinct. (Already known by Erdös via diff proof.)

Next Step: Finite version: Can use Finite Can Ramsey to prove the following: For every set of n points in the plane there is a subset of size $\Omega(\log n)$ where all distances are distinct. (Much better is known.)

INITIAL MOTIVATION ABANDONED

INITIAL MOTIVATION ABANDONED

1. Dumped Ramsey**!** Added co-authors**!** Got **new** results!

INITIAL MOTIVATION ABANDONED

1. Dumped Ramsey**!** Added co-authors**!** Got **new** results!
2. What about **Area**? If there are n points in $\mathbb{R}^2$ want large subset so that all areas are distinct.

INITIAL MOTIVATION ABANDONED

1. Dumped Ramsey**!** Added co-authors**!** Got **new** results!
2. What about **Area**? If there are n points in $\mathbb{R}^2$ want large subset so that all areas are distinct.
3. More general question: n points in $\mathbb{R}^d$ and looking for all a-volumes to be different. (This question seems to be new.)

EXAMPLES with DISTANCES

The following is an **EXAMPLE** of the kind of theorems we will be talking about.

If there are n points in $\mathbb{R}^2$ *then there is a subset of size* $\Omega(n^{1/3})$ *with all distances between points* **DIFF**.

EXAMPLES with AREAS

If there are n points in $\mathbb{R}^2$ then there is a subset of size $\Omega(n^{1/5})$ with all triangle areas **DIFF**.

EXAMPLES with AREAS

If there are n points in $\mathbb{R}^2$ *then there is a subset of size* $\Omega(n^{1/5})$ *with all triangle areas* **DIFF**.

FALSE: Take n points on a LINE. All triangle areas are 0.

EXAMPLES with AREAS

If there are n points in $\mathbb{R}^2$ then there is a subset of size $\Omega(n^{1/5})$ with all triangle areas **DIFF**.

FALSE: Take n points on a LINE. All triangle areas are 0.

We state theorems in **no three collinear** form to get around this issue.

Maximal Rainbow Sets

Def: A **(2)-Rainbow Set** is a set of points in $\mathbb{R}^d$ where all of the distances are distinct. Also called a **dist-rainbow.**

Maximal Rainbow Sets

Def: A **(2)-Rainbow Set** is a set of points in $\mathbb{R}^d$ where all of the distances are distinct. Also called a **dist-rainbow.**
Def: A **3-Rainbow Set** is a set of points in $\mathbb{R}^d$ where all nonzero areas of triangles are distinct. Also called an **area-rainbow.**

Maximal Rainbow Sets

Def: A **(2)-Rainbow Set** is a set of points in $\mathbb{R}^d$ where all of the distances are distinct. Also called a **dist-rainbow.**

Def: A **3-Rainbow Set** is a set of points in $\mathbb{R}^d$ where all nonzero areas of triangles are distinct. Also called an **area-rainbow.**

Def: An **a-Rainbow Set** is a set of points in $\mathbb{R}^d$ where all nonzero a-volumes are distinct. An a-volume is the volume enclosed by a points. Also called a **vol-rainbow.**

Maximal Rainbow Sets

Def: A **(2)-Rainbow Set** is a set of points in $\mathbb{R}^d$ where all of the distances are distinct. Also called a **dist-rainbow.**
Def: A **3-Rainbow Set** is a set of points in $\mathbb{R}^d$ where all nonzero areas of triangles are distinct. Also called an **area-rainbow.**
Def: An **a-Rainbow Set** is a set of points in $\mathbb{R}^d$ where all nonzero a-volumes are distinct. An a-volume is the volume enclosed by a points. Also called a **vol-rainbow.**

Def: Let $X \subseteq \mathbb{R}^d$. A **Maximal Rainbow Set** is a rainbow set $Y \subseteq X$ such that if any more points of X are added then it STOPS being a rainbow set.

Maximal Rainbow Sets

Def: A **(2)-Rainbow Set** is a set of points in $\mathbb{R}^d$ where all of the distances are distinct. Also called a **dist-rainbow.**
Def: A **3-Rainbow Set** is a set of points in $\mathbb{R}^d$ where all nonzero areas of triangles are distinct. Also called an **area-rainbow.**
Def: An **a-Rainbow Set** is a set of points in $\mathbb{R}^d$ where all nonzero a-volumes are distinct. An a-volume is the volume enclosed by a points. Also called a **vol-rainbow.**

Def: Let $X \subseteq \mathbb{R}^d$. A **Maximal Rainbow Set** is a rainbow set $Y \subseteq X$ such that if any more points of X are added then it STOPS being a rainbow set.
Def: Let $X \subseteq \mathbb{R}^d$. An **a-Maximal Rainbow Set** is a a-rainbow set $Y \subseteq X$ such that if any more points of X are added then it STOPS being an a-rainbow set.

Easy Lemma

Lemma If there is a MAP from X to Y that is $\leq c$-to-1 then
$|Y| \geq |X|/c$.
We will call this LEMMA.

The $d = 1$ Case

Thm: For all $X \subseteq \mathbb{R}^1$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/3})$.

The $d = 1$ Case

Thm: For all $X \subseteq \mathbb{R}^1$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/3})$.
Proof: Let M be a **MAXIMAL DIST-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

The $d = 1$ Case

Thm: For all $X \subseteq \mathbb{R}^1$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/3})$.
Proof: Let M be a **MAXIMAL DIST-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$, or
- $(\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|]$.

The $d = 1$ Case

Thm: For all $X \subseteq \mathbb{R}^1$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/3})$.
Proof: Let M be a **MAXIMAL DIST-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$, or
- $(\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|]$.

f maps an element of $X - M$ to **reason** $x \notin M$.

The $d = 1$ Case

Thm: For all $X \subseteq \mathbb{R}^1$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/3})$.
Proof: Let M be a **MAXIMAL DIST-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$, or
- $(\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|]$.

f maps an element of $X - M$ to **reason** $x \notin M$.
$f : X - M \to \binom{M}{2} \cup M \times \binom{M}{2}$

The $d = 1$ Case

Thm: For all $X \subseteq \mathbb{R}^1$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/3})$.
Proof: Let M be a **MAXIMAL DIST-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$, or
- $(\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|]$.

f maps an element of $X - M$ to **reason** $x \notin M$.
$f : X - M \rightarrow \binom{M}{2} \cup M \times \binom{M}{2}$
What is $f^{-1}(\{x_1, x_2\})$?

The $d = 1$ Case

Thm: For all $X \subseteq \mathbb{R}^1$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/3})$.
Proof: Let M be a **MAXIMAL DIST-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$, or
- $(\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|]$.

f maps an element of $X - M$ to **reason** $x \notin M$.
$f : X - M \rightarrow \binom{M}{2} \cup M \times \binom{M}{2}$
What is $f^{-1}(\{x_1, x_2\})$? It's ≤ 1 POINT.

The $d = 1$ Case

Thm: For all $X \subseteq \mathbb{R}^1$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/3})$.
Proof: Let M be a **MAXIMAL DIST-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$, or
- $(\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|]$.

f maps an element of $X - M$ to **reason** $x \notin M$.
$f : X - M \rightarrow \binom{M}{2} \cup M \times \binom{M}{2}$
What is $f^{-1}(\{x_1, x_2\})$? It's ≤ 1 POINT.
What is $f^{-1}(x_1, \{x_2, x_3\})$?

The $d = 1$ Case

Thm: For all $X \subseteq \mathbb{R}^1$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/3})$.
Proof: Let M be a **MAXIMAL DIST-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$, or
- $(\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|]$.

f maps an element of $X - M$ to **reason** $x \notin M$.
$f : X - M \rightarrow \binom{M}{2} \cup M \times \binom{M}{2}$
What is $f^{-1}(\{x_1, x_2\})$? It's ≤ 1 POINT.
What is $f^{-1}(x_1, \{x_2, x_3\})$? It's ≤ 2 POINTS.

The $d = 1$ Case

Thm: For all $X \subseteq \mathbb{R}^1$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/3})$.
Proof: Let M be a **MAXIMAL DIST-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$, or
- $(\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|]$.

f maps an element of $X - M$ to **reason** $x \notin M$.
$f : X - M \rightarrow \binom{M}{2} \cup M \times \binom{M}{2}$
What is $f^{-1}(\{x_1, x_2\})$? It's ≤ 1 POINT.
What is $f^{-1}(x_1, \{x_2, x_3\})$? It's ≤ 2 POINTS.

$f : X - M \rightarrow \binom{M}{2} \cup M \times \binom{M}{2}$ is $\le$ 2-to-1.

The $d = 1$ Case- Cont

$f : X - M \to \binom{M}{2} \cup M \times \binom{M}{2}$ is $\leq$ 2-to-1.

The $d = 1$ Case- Cont

$f : X - M \to \binom{M}{2} \cup M \times \binom{M}{2}$ is $\leq$ 2-to-1.

Case 1: $|M| \geq n^{1/3}$ DONE!

The $d = 1$ Case- Cont

$f : X - M \to \binom{M}{2} \cup M \times \binom{M}{2}$ is $\le$ 2-to-1.

Case 1: $|M| \ge n^{1/3}$ DONE!

Case 2: $|M| \le n^{1/3}$. So $|X - M| = \Theta(|X|)$. By LEMMA

$$\begin{aligned} |\binom{M}{2} + M \times \binom{M}{2}| &\ge 0.5|X - M| = \Omega(|X|) = \Omega(n) \\ M &\ge \Omega(n^{1/3}) \end{aligned}$$

On Circle

Thm: For all $X \subseteq \mathbb{S}^1$ (the circle) of size n there exists a dist-rainbow subset of size $\Omega(n^{1/3})$.
Proof: Use **MAXIMAL DIST-RAINBOW SET**. Similar Proof.

Better is known

Better is known: In 1975 Komlos, Sulyok, Szemeredi showed:
Thm: For all $X \subseteq \mathbb{S}^1$ or $\mathbb{R}^1$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/2})$.

Better is known

Better is known: In 1975 Komlos, Sulyok, Szemeredi showed:
Thm: For all $X \subseteq \mathbb{S}^1$ or $\mathbb{R}^1$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/2})$.

This is optimal in $\mathbb{S}^1$ and $\mathbb{R}^1$

Better is known

Better is known: In 1975 Komlos, Sulyok, Szemeredi showed:
Thm: For all $X \subseteq \mathbb{S}^1$ or $\mathbb{R}^1$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/2})$.

This is optimal in $\mathbb{S}^1$ and $\mathbb{R}^1$
Thm: If $X = \{1, \ldots, n\}$ then the largest dist-rainbow subset is of size $\leq (1 + o(1))n^{1/2}$.

The $d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/6})$.
Proof: Let M be a **MAXIMAL DIST-RAINBOW SET.**

The $d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/6})$.
Proof: Let M be a **MAXIMAL DIST-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

The $d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/6})$.
Proof: Let M be a **MAXIMAL DIST-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$.

The $d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/6})$.
Proof: Let M be a **MAXIMAL DIST-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$.
- $(\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|]$.

f maps an element of $X - M$ to **reason** $x \notin M$.
$f : X - M \to \binom{M}{2} \cup M \times \binom{M}{2}$
What is $f^{-1}(\{x_1, x_2\})$?

The $d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/6})$.
Proof: Let M be a **MAXIMAL DIST-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$.
- $(\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|]$.

f maps an element of $X - M$ to **reason** $x \notin M$.
$f : X - M \rightarrow \binom{M}{2} \cup M \times \binom{M}{2}$
What is $f^{-1}(\{x_1, x_2\})$? Lies on LINE.

The $d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/6})$.
Proof: Let M be a **MAXIMAL DIST-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$.
- $(\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|]$.

f maps an element of $X - M$ to **reason** $x \notin M$.
$f : X - M \rightarrow \binom{M}{2} \cup M \times \binom{M}{2}$
What is $f^{-1}(\{x_1, x_2\})$? Lies on LINE.
What is $f^{-1}(x_1, \{x_2, x_3\})$?

The $d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/6})$.
Proof: Let M be a **MAXIMAL DIST-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$.
- $(\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|]$.

f maps an element of $X - M$ to **reason** $x \notin M$.
$f : X - M \to \binom{M}{2} \cup M \times \binom{M}{2}$
What is $f^{-1}(\{x_1, x_2\})$? Lies on LINE.
What is $f^{-1}(x_1, \{x_2, x_3\})$? Lies on CIRCLE.

The $d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/6})$.
Proof: Let M be a **MAXIMAL DIST-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2 \in M)[|x - x_1| = |x - x_2|]$.
- $(\exists x_1, x_2, x_3 \in M)[|x - x_1| = |x_2 - x_3|]$.

f maps an element of $X - M$ to **reason** $x \notin M$.
$f : X - M \rightarrow \binom{M}{2} \cup M \times \binom{M}{2}$
What is $f^{-1}(\{x_1, x_2\})$? Lies on LINE.
What is $f^{-1}(x_1, \{x_2, x_3\})$? Lies on CIRCLE.
All INVERSE IMG's lie on LINES or CIRCLES.

The $d = 2$ Case- Cont

$f : X - M \to \binom{M}{2} \cup M \times \binom{M}{2}$
All INVERSE IMG's lie on LINES or CIRCLES. δ TBD.
Cases 1 and 2 induct into line and circle case.
Case 1: $(\exists x_1, x_2)[(f^{-1}(\{x_1, x_2\})| \geq n^{\delta}]$.
$\geq n^{\delta}$ points on a line, so rainbow set size $\geq \Omega(n^{\delta/3})$.

The $d = 2$ Case- Cont

$f : X - M \rightarrow \binom{M}{2} \cup M \times \binom{M}{2}$
All INVERSE IMG's lie on LINES or CIRCLES. δ TBD.
Cases 1 and 2 induct into line and circle case.
Case 1: $(\exists x_1, x_2)[(f^{-1}(\{x_1, x_2\})| \geq n^\delta]$.
$\geq n^\delta$ points on a line, so rainbow set size $\geq \Omega(n^{\delta/3})$.
Case 2: $(\exists x_1, x_2, x_3)[|f^{-1}(\{x_1, \{x_2, x_3\}\})| \geq n^\delta]$.
$\geq n^\delta$ points on a circle, so rainbow set size $\geq \Omega(n^{\delta/3})$.

The $d = 2$ Case- Cont

$f : X - M \rightarrow \binom{M}{2} \cup M \times \binom{M}{2}$

All INVERSE IMG's lie on LINES or CIRCLES. δ TBD.

Cases 1 and 2 induct into line and circle case.

Case 1: $(\exists x_1, x_2)[(f^{-1}(\{x_1, x_2\})| \geq n^{\delta}]$.

$\geq n^{\delta}$ points on a line, so rainbow set size $\geq \Omega(n^{\delta/3})$.

Case 2: $(\exists x_1, x_2, x_3)[|f^{-1}(\{x_1, \{x_2, x_3\}\})| \geq n^{\delta}]$.

$\geq n^{\delta}$ points on a circle, so rainbow set size $\geq \Omega(n^{\delta/3})$.

Case 3: $|M| \geq n^{1/6}$ DONE!

The $d = 2$ Case- Cont

$f : X - M \rightarrow \binom{M}{2} \cup M \times \binom{M}{2}$
All INVERSE IMG's lie on LINES or CIRCLES. δ TBD.
Cases 1 and 2 induct into line and circle case.

Case 1: $(\exists x_1, x_2)[(f^{-1}(\{x_1, x_2\})| \geq n^\delta]$.
$\geq n^\delta$ points on a line, so rainbow set size $\geq \Omega(n^{\delta/3})$.

Case 2: $(\exists x_1, x_2, x_3)[|f^{-1}(\{x_1, \{x_2, x_3\}\})| \geq n^\delta]$.
$\geq n^\delta$ points on a circle, so rainbow set size $\geq \Omega(n^{\delta/3})$.

Case 3: $|M| \geq n^{1/6}$ DONE!

Case 4: Map is $\leq n^\delta$-to-1 AND $|X - M| = \Theta(|X|)$. By LEMMA

The $d = 2$ Case- Cont

$f : X - M \rightarrow \binom{M}{2} \cup M \times \binom{M}{2}$
All INVERSE IMG's lie on LINES or CIRCLES. δ TBD.
Cases 1 and 2 induct into line and circle case.
Case 1: $(\exists x_1, x_2)[(f^{-1}(\{x_1, x_2\})| \ge n^{\delta}]$.
$\ge n^{\delta}$ points on a line, so rainbow set size $\ge \Omega(n^{\delta/3})$.
Case 2: $(\exists x_1, x_2, x_3)[|f^{-1}(\{x_1, \{x_2, x_3\}\})| \ge n^{\delta}]$.
$\ge n^{\delta}$ points on a circle, so rainbow set size $\ge \Omega(n^{\delta/3})$.
Case 3: $|M| \ge n^{1/6}$ DONE!
Case 4: Map is $\le n^{\delta}$-to-1 AND $|X - M| = \Theta(|X|)$. By LEMMA

$$\begin{aligned} |\tbinom{M}{2} \cup M \times \tbinom{M}{2}| &\ge n/n^{\delta} = n^{1-\delta} \\ |M| &\ge \Omega(n^{(1-\delta)/3}) \end{aligned}$$

The $d = 2$ Case- Cont

$f : X - M \rightarrow \binom{M}{2} \cup M \times \binom{M}{2}$
All INVERSE IMG's lie on LINES or CIRCLES. δ TBD.
Cases 1 and 2 induct into line and circle case.
Case 1: $(\exists x_1, x_2)[(f^{-1}(\{x_1, x_2\})| \geq n^{\delta}]$.
$\geq n^{\delta}$ points on a line, so rainbow set size $\geq \Omega(n^{\delta/3})$.
Case 2: $(\exists x_1, x_2, x_3)[|f^{-1}(\{x_1, \{x_2, x_3\}\})| \geq n^{\delta}]$.
$\geq n^{\delta}$ points on a circle, so rainbow set size $\geq \Omega(n^{\delta/3})$.
Case 3: $|M| \geq n^{1/6}$ DONE!
Case 4: Map is $\leq n^{\delta}$-to-1 AND $|X - M| = \Theta(|X|)$. By LEMMA

$$\begin{aligned} |\binom{M}{2} \cup M \times \binom{M}{2}| &\geq n/n^{\delta} = n^{1-\delta} \\ |M| &\geq \Omega(n^{(1-\delta)/3}) \end{aligned}$$

Set $\delta/3 = (1 - \delta)/3$. $\delta = 1/2$. Get $\Omega(n^{1/6})$.

On Sphere

Thm: For all $X \subseteq \mathbb{S}^2$ (surface of sphere) of size n there exists a dist-rainbow subset of size $\Omega(n^{1/6})$.
Proof: Use **MAXIMAL DIST-RAINBOW SET**. Similar Proof.

On Sphere

Thm: For all $X \subseteq \mathbb{S}^2$ (surface of sphere) of size n there exists a dist-rainbow subset of size $\Omega(n^{1/6})$.
Proof: Use **MAXIMAL DIST-RAINBOW SET**. Similar Proof.

Note: Better is known: Charalambides showed $\Omega(n^{1/3})$.

General d Case

Thm:
For all $X \subseteq \mathbb{R}^d$ of size n $\exists$ dist-rainbow subset of size $\Omega(n^{1/3d})$.
For all $X \subseteq \mathbb{S}^d$ of size n $\exists$ dist-rainbow subset of size $\Omega(n^{1/3d})$.

General d Case

Thm:
For all $X \subseteq \mathbb{R}^d$ of size n $\exists$ dist-rainbow subset of size $\Omega(n^{1/3d})$.
For all $X \subseteq \mathbb{S}^d$ of size n $\exists$ dist-rainbow subset of size $\Omega(n^{1/3d})$.

Proof: Use **MAXIMAL DIST-RAINBOW SET** and induction.
Need result on $\mathbb{S}^d$ and $\mathbb{R}^d$ to get result for $\mathbb{S}^{d+1}$ and $\mathbb{R}^{d+1}$.

General d Case

Thm:
For all $X \subseteq \mathbb{R}^d$ of size n $\exists$ dist-rainbow subset of size $\Omega(n^{1/3d})$.
For all $X \subseteq \mathbb{S}^d$ of size n $\exists$ dist-rainbow subset of size $\Omega(n^{1/3d})$.

Proof: Use **MAXIMAL DIST-RAINBOW SET** and induction.
Need result on $\mathbb{S}^d$ and $\mathbb{R}^d$ to get result for $\mathbb{S}^{d+1}$ and $\mathbb{R}^{d+1}$.

Note: Better is known. In 1995 Thiele showed $\Omega(n^{1/(3d-2)})$. But WE improved that!

General d Case- Much Better

Thm: For all $d \geq 2$, for all $X \subseteq \mathbb{R}^d$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/(3d-3)}(\log n)^{\frac{1}{3}-\frac{2}{3d-3}})$.

General d Case- Much Better

Thm: For all $d \geq 2$, for all $X \subseteq \mathbb{R}^d$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/(3d-3)}(\log n)^{\frac{1}{3}-\frac{2}{3d-3}})$.

Proof: Use **VARIANT ON MAX DIST-RAINBOW SET**

d	$n^{1/3d}$	$n^{1/(3d-3)}(\log n)^{\frac{1}{3}-\frac{2}{3d-3}}$
1	$n^{1/3}$	$--$
2	$n^{1/6}$	$n^{1/3}(\log n)^{-1/3}$
3	$n^{1/9}$	$n^{1/6}(\log n)^{0}$
4	$n^{1/12}$	$n^{1/9}(\log n)^{1/12}$
5	$n^{1/15}$	$n^{1/12}(\log n)^{1/6}$
6	$n^{1/18}$	$n^{1/15}(\log n)^{1/5}$

General d Case- Much Better

Thm: For all $d \geq 2$, for all $X \subseteq \mathbb{R}^d$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/(3d-3)}(\log n)^{\frac{1}{3}-\frac{2}{3d-3}})$.

Proof: Use **VARIANT ON MAX DIST-RAINBOW SET**

d	$n^{1/3d}$	$n^{1/(3d-3)}(\log n)^{\frac{1}{3}-\frac{2}{3d-3}}$
1	$n^{1/3}$	$--$
2	$n^{1/6}$	$n^{1/3}(\log n)^{-1/3}$
3	$n^{1/9}$	$n^{1/6}(\log n)^0$
4	$n^{1/12}$	$n^{1/9}(\log n)^{1/12}$
5	$n^{1/15}$	$n^{1/12}(\log n)^{1/6}$
6	$n^{1/18}$	$n^{1/15}(\log n)^{1/5}$

Can we do better? Best we can hope for is roughly $n^{1/d}$.

Area-$d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.

Area-$d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.
Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Area-$d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.
Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

Area-$d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.
Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.

Area-$d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.
Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.

Area-$d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x_3, x_4, x_5)]$.

Area-$d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.
Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x_3, x_4, x_5)]$.

f maps an element of $X - M$ to **reason** $x \notin M$.

Area-$d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.
Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x_3, x_4, x_5)]$.

f maps an element of $X - M$ to **reason** $x \notin M$.
$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$. Recall that

Area-$d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.
Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x_3, x_4, x_5)]$.

f maps an element of $X - M$ to **reason** $x \notin M$.
$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$. Recall that
What is $f^{-1}(\{x_1, x_2\}, \{x_1, x_3\})$? SEE NEXT SLIDE FOR GEOM LEMMA.

Lemma On Area

Lemma: Let L_1 and L_2 be lines in R^2.

$$\{p : \mathrm{AREA}(L_1, p) = \mathrm{AREA}(L_2, p)\}$$

is a line.

Lemma On Area

Lemma: Let L_1 and L_2 be lines in R^2.

$$\{p : \mathrm{AREA}(L_1, p) = \mathrm{AREA}(L_2, p)\}$$

is a line.

Sketch: $\mathrm{AREA}(L_1, p) = \mathrm{AREA}(L_2, p)$ iff $|L_1| \times |L_1 - p| = |L_2| \times |L_2 - p|$ iff $\frac{|L_1 - p|}{|L_2 - p|} = \frac{|L_1|}{|L_2|}$. This is a line.

(Reboot) Area-$d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.

(Reboot) Area-$d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

(Reboot) Area-$d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Let $x \in X - M$. WHY IS x NOT IN M!? Either

(Reboot) Area-$d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.

(Reboot) Area-$d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.

(Reboot) Area-$d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x_3, x_4, x_5)]$.

(Reboot) Area-$d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x_3, x_4, x_5)]$.

f maps an element of $X - M$ to **reason** $x \notin M$.

(Reboot) Area-$d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x_3, x_4, x_5)]$.

f maps an element of $X - M$ to **reason** $x \notin M$.

$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$. Recall that

(Reboot) Area-$d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x_3, x_4, x_5)]$.

f maps an element of $X - M$ to **reason** $x \notin M$.

$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$. Recall that

What is $f^{-1}(\{x_1, x_2\}, \{x_1, x_3\})$?

(Reboot) Area-$d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x_3, x_4, x_5)]$.

f maps an element of $X - M$ to **reason** $x \notin M$.

$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$. Recall that

What is $f^{-1}(\{x_1, x_2\}, \{x_1, x_3\})$? By Lemma all points on it are on a line- so ≤ 2 points. FINITE.

(Reboot) Area-$d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x_3, x_4, x_5)]$.

f maps an element of $X - M$ to **reason** $x \notin M$.

$f : X - M \to \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$. Recall that

What is $f^{-1}(\{x_1, x_2\}, \{x_1, x_3\})$? By Lemma all points on it are on a line- so ≤ 2 points. FINITE.

What is $f^{-1}(\{x_1, x_2\}, \{x_3, x_4\})$?

(Reboot) Area-$d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x_3, x_4, x_5)]$.

f maps an element of $X - M$ to **reason** $x \notin M$.

$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$. Recall that

What is $f^{-1}(\{x_1, x_2\}, \{x_1, x_3\})$? By Lemma all points on it are on a line- so ≤ 2 points. FINITE.

What is $f^{-1}(\{x_1, x_2\}, \{x_3, x_4\})$? By Lemma all points on it are on a line- so ≤ 2 points. FINITE.

(Reboot) Area-$d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x_3, x_4, x_5)]$.

f maps an element of $X - M$ to **reason** $x \notin M$.

$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$. Recall that

What is $f^{-1}(\{x_1, x_2\}, \{x_1, x_3\})$? By Lemma all points on it are on a line- so ≤ 2 points. FINITE.

What is $f^{-1}(\{x_1, x_2\}, \{x_3, x_4\})$? By Lemma all points on it are on a line- so ≤ 2 points. FINITE.

What is $f^{-1}(\{x_1, x_2\}, \{x_3, x_4, x_5\})$?

(Reboot) Area-$d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n, no three colinear, $\exists$ area-rainbow set of size $\Omega(n^{1/5})$.

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x_3, x_4, x_5)]$.

f maps an element of $X - M$ to **reason** $x \notin M$.

$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$. Recall that

What is $f^{-1}(\{x_1, x_2\}, \{x_1, x_3\})$? By Lemma all points on it are on a line- so ≤ 2 points. FINITE.

What is $f^{-1}(\{x_1, x_2\}, \{x_3, x_4\})$? By Lemma all points on it are on a line- so ≤ 2 points. FINITE.

What is $f^{-1}(\{x_1, x_2\}, \{x_3, x_4, x_5\})$? By Lemma all points on it are on a line- so ≤ 2 points. FINITE.

$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$ FINITE-to-1.

Area $d = 2$ Case- Cont

$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$ is FINITE-to-1.
Case 1: $|M| \geq n^{1/5}$ DONE!

Case 2: $|M| \leq n^{1/5}$. Then $|X - M| = \Theta(|X|)$. Since MAP is finite-to-1, by LEMMA

$$
\begin{aligned}
|\binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}| &\geq \Omega(|X - M|) = \Omega(|X|) = \Omega(n) \\
|M| &\geq \Omega(n^{1/5})
\end{aligned}
$$

Volume $d = 3$

Thm: For all $X \subseteq \mathbb{R}^3$ of size n, no four on a plane, there exists Vol-rainbow set of size $\Omega(n^\delta)$. (δ TBD)
Similar. Left for the reader.

KEY to These Proofs

KEY to These Proofs

1. Used **MAXIMAL a-RAINBOW SET** M.

KEY to These Proofs

1. Used **MAXIMAL a-RAINBOW SET** M.
2. Used Map f from $x \in X - M$ to the reason x is NOT in M.

KEY to These Proofs

1. Used **MAXIMAL a-RAINBOW SET** M.
2. Used Map f from $x \in X - M$ to the reason x is NOT in M.
3. Looked at **INVERSE IMAGES** of that map.

KEY to These Proofs

1. Used **MAXIMAL a-RAINBOW SET** M.
2. Used Map f from $x \in X - M$ to the reason x is NOT in M.
3. Looked at **INVERSE IMAGES** of that map.
4. Either:
   All INVERSE IMG's are small, so use LEMMA.
   OR
   Some INVERSE IMG's are large subsets of $\mathbb{R}^d$ or $\mathbb{S}^d$, so induct.

Area-$d = 3$ Case

Thm: For all $X \subseteq \mathbb{R}^3$ of size n, no three colinear, there exists Area-rainbow set of size $\Omega(n^{\delta})$. (δ TBD)

Area-$d = 3$ Case

Thm: For all $X \subseteq \mathbb{R}^3$ of size n, no three colinear, there exists Area-rainbow set of size $\Omega(n^{\delta})$. (δ TBD)
Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Area-$d = 3$ Case

Thm: For all $X \subseteq \mathbb{R}^3$ of size n, no three colinear, there exists Area-rainbow set of size $\Omega(n^\delta)$. (δ TBD)
Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

Area-$d = 3$ Case

Thm: For all $X \subseteq \mathbb{R}^3$ of size n, no three colinear, there exists Area-rainbow set of size $\Omega(n^\delta)$. (δ TBD)

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.

Area-$d = 3$ Case

Thm: For all $X \subseteq \mathbb{R}^3$ of size n, no three colinear, there exists Area-rainbow set of size $\Omega(n^\delta)$. (δ TBD)

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.

Area-$d = 3$ Case

Thm: For all $X \subseteq \mathbb{R}^3$ of size n, no three colinear, there exists Area-rainbow set of size $\Omega(n^\delta)$. (δ TBD)

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x_3, x_4, x_5)]$.

Area-$d = 3$ Case

Thm: For all $X \subseteq \mathbb{R}^3$ of size n, no three colinear, there exists Area-rainbow set of size $\Omega(n^\delta)$. (δ TBD)

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x_3, x_4, x_5)]$.

f maps an element of $X - M$ to **reason** $x \notin M$.

Area-$d = 3$ Case

Thm: For all $X \subseteq \mathbb{R}^3$ of size n, no three colinear, there exists Area-rainbow set of size $\Omega(n^\delta)$. (δ TBD)

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x_3, x_4, x_5)]$.

f maps an element of $X - M$ to **reason** $x \notin M$.

$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$.

Area-$d = 3$ Case

Thm: For all $X \subseteq \mathbb{R}^3$ of size n, no three colinear, there exists Area-rainbow set of size $\Omega(n^\delta)$. (δ TBD)

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x_3, x_4, x_5)]$.

f maps an element of $X - M$ to **reason** $x \notin M$.

$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$.

What is $f^{-1}(\{\{x_1, x_2\}, \{x_1, x_3\}\})$?

Area-$d = 3$ Case

Thm: For all $X \subseteq \mathbb{R}^3$ of size n, no three colinear, there exists Area-rainbow set of size $\Omega(n^\delta)$. (δ TBD)

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x_3, x_4, x_5)]$.

f maps an element of $X - M$ to **reason** $x \notin M$.
$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$.
What is $f^{-1}(\{\{x_1, x_2\}, \{x_1, x_3\}\})$? THIS IS HARD!

Area-$d = 3$ Case

Thm: For all $X \subseteq \mathbb{R}^3$ of size n, no three colinear, there exists Area-rainbow set of size $\Omega(n^\delta)$. (δ TBD)

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x_3, x_4, x_5)]$.

f maps an element of $X - M$ to **reason** $x \notin M$.

$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$.

What is $f^{-1}(\{\{x_1, x_2\}, \{x_1, x_3\}\})$? THIS IS HARD!

What is $f^{-1}(\{x_1, x_2\}, \{x_3, x_4\})$?

Area-$d = 3$ Case

Thm: For all $X \subseteq \mathbb{R}^3$ of size n, no three colinear, there exists Area-rainbow set of size $\Omega(n^\delta)$. (δ TBD)

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**
Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x_3, x_4, x_5)]$.

f maps an element of $X - M$ to **reason** $x \notin M$.
$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$.
What is $f^{-1}(\{\{x_1, x_2\}, \{x_1, x_3\})$? THIS IS HARD!

What is $f^{-1}(\{x_1, x_2\}, \{x_3, x_4\})$? THIS IS HARD!

Area-$d = 3$ Case

Thm: For all $X \subseteq \mathbb{R}^3$ of size n, no three colinear, there exists Area-rainbow set of size $\Omega(n^\delta)$. (δ TBD)

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x_3, x_4, x_5)]$.

f maps an element of $X - M$ to **reason** $x \notin M$.

$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$.

What is $f^{-1}(\{\{x_1, x_2\}, \{x_1, x_3\})$? THIS IS HARD!

What is $f^{-1}(\{x_1, x_2\}, \{x_3, x_4\})$? THIS IS HARD!

What is $f^{-1}(\{x_1, x_2\}, \{x_3, x_4, x_5\})$?

Area-$d = 3$ Case

Thm: For all $X \subseteq \mathbb{R}^3$ of size n, no three colinear, there exists Area-rainbow set of size $\Omega(n^\delta)$. (δ TBD)

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x_3, x_4, x_5)]$.

f maps an element of $X - M$ to **reason** $x \notin M$.

$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$.

What is $f^{-1}(\{\{x_1, x_2\}, \{x_1, x_3\}\})$? THIS IS HARD!

What is $f^{-1}(\{x_1, x_2\}, \{x_3, x_4\})$? THIS IS HARD!

What is $f^{-1}(\{x_1, x_2\}, \{x_3, x_4, x_5\})$? THIS IS HARD!

Area-$d = 3$ Case

Thm: For all $X \subseteq \mathbb{R}^3$ of size n, no three colinear, there exists Area-rainbow set of size $\Omega(n^\delta)$. (δ TBD)

Proof: Let M be a **MAXIMAL AREA-RAINBOW SET.**

Let $x \in X - M$. WHY IS x NOT IN M!? Either

- $(\exists x_1, x_2, x_3 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_1, x_3)]$.
- $(\exists x_1, x_2, x_3, x_4 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)]$.
- $(\exists x_1, x_2, x_3, x_4, x_5 \in M)[\mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x_3, x_4, x_5)]$.

f maps an element of $X - M$ to **reason** $x \notin M$.

$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$.

What is $f^{-1}(\{\{x_1, x_2\}, \{x_1, x_3\}\})$? THIS IS HARD!

What is $f^{-1}(\{x_1, x_2\}, \{x_3, x_4\})$? THIS IS HARD!

What is $f^{-1}(\{x_1, x_2\}, \{x_3, x_4, x_5\})$? THIS IS HARD!

What to do?

WHAT CHANGED?

Why is this proof harder?
KEY statement about prior proof:

WHAT CHANGED?

Why is this proof harder?
KEY statement about prior proof:

1. If INVERSE IMG's are all finite so M is large.

WHAT CHANGED?

Why is this proof harder?
KEY statement about prior proof:

1. If INVERSE IMG's are all finite so M is large.
2. If INVERSE IMG's are subsets of $\mathbb{R}^d$ or $\mathbb{S}^d$ then induct.

WHAT CHANGED?

Why is this proof harder?
KEY statement about prior proof:

1. If INVERSE IMG's are all finite so M is large.
2. If INVERSE IMG's are subsets of $\mathbb{R}^d$ or $\mathbb{S}^d$ then induct.

KEY: We cared about $X \subseteq \mathbb{R}^d$ but had to work with $\mathbb{S}^d$ as well.
NOW we will have to work with more complicated objects.

What Do Inverse Images Look Like?

$$\{x : \mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)\} =$$

$$\{x : |\mathrm{DET}(x, x_1, x_2)| = |\mathrm{DET}(x, x_3, x_4)|\}.$$

What Do Inverse Images Look Like?

$$\{x : \mathrm{AREA}(x, x_1, x_2) = \mathrm{AREA}(x, x_3, x_4)\} =$$

$$\{x : |\mathrm{DET}(x, x_1, x_2)| = |\mathrm{DET}(x, x_3, x_4)|\}.$$

Def: (Informally) An **Algebraic Variety in $\mathbb{R}^{\mathbf{d}}$** is a set of points in $\mathbb{R}^d$ that satisfy a polynomial equation in d variables.

General Thm

Thm Let $2 \leq a \leq d+1$. Let $r \in \mathbb{N}$. For all varieties V of dim d and degree r for all sets of n points on V there exists an a-rainbow set of size $\Omega(n^{1/(2a-1)d})$.

General Thm

Thm Let $2 \leq a \leq d+1$. Let $r \in \mathbb{N}$. For all varieties V of dim d and degree r for all sets of n points on V there exists an a-rainbow set of size $\Omega(n^{1/(2a-1)d})$.

Corollary Let $2 \leq a \leq d+1$. For all $X \subseteq \mathbb{R}^d$ of size n there exists an a-rainbow set of size $\Omega(n^{1/(2a-1)d})$.

General Thm

Thm Let $2 \le a \le d+1$. Let $r \in \mathbb{N}$. For all varieties V of dim d and degree r for all sets of n points on V there exists an a-rainbow set of size $\Omega(n^{1/(2a-1)d})$.

Corollary Let $2 \le a \le d+1$. For all $X \subseteq \mathbb{R}^d$ of size n there exists an a-rainbow set of size $\Omega(n^{1/(2a-1)d})$.

Corollary For all $X \subseteq \mathbb{R}^d$ of size n there exists a 2-rainbow set (dist. distances) of size $\Omega(n^{1/3d})$.

General Thm

Thm Let $2 \le a \le d+1$. Let $r \in \mathbb{N}$. For all varieties V of dim d and degree r for all sets of n points on V there exists an a-rainbow set of size $\Omega(n^{1/(2a-1)d})$.

Corollary Let $2 \le a \le d+1$. For all $X \subseteq \mathbb{R}^d$ of size n there exists an a-rainbow set of size $\Omega(n^{1/(2a-1)d})$.

Corollary For all $X \subseteq \mathbb{R}^d$ of size n there exists a 2-rainbow set (dist. distances) of size $\Omega(n^{1/3d})$.

Corollary For all $X \subseteq \mathbb{R}^d$ of size n there is a 3-rainbow set (dist. areas) of size $\Omega(n^{1/5d})$.

General Thm

Thm Let $2 \leq a \leq d+1$. Let $r \in \mathbb{N}$. For all varieties V of dim d and degree r for all sets of n points on V there exists an a-rainbow set of size $\Omega(n^{1/(2a-1)d})$.

Corollary Let $2 \leq a \leq d+1$. For all $X \subseteq \mathbb{R}^d$ of size n there exists an a-rainbow set of size $\Omega(n^{1/(2a-1)d})$.

Corollary For all $X \subseteq \mathbb{R}^d$ of size n there exists a 2-rainbow set (dist. distances) of size $\Omega(n^{1/3d})$.

Corollary For all $X \subseteq \mathbb{R}^d$ of size n there is a 3-rainbow set (dist. areas) of size $\Omega(n^{1/5d})$.

Corollary For all $X \subseteq \mathbb{R}^d$ of size n there is a 4-rainbow set (dist. volumes) of size $\Omega(n^{1/7d})$.

Comments on the Proof

General Thm

Thm Let $2 \le a \le d+1$. Let $r \in \mathbb{N}$. For all varieties V of dim d and degree r for all sets of n points on V there exists an a-rainbow set of size $\Omega(n^{1/(2a-1)d})$.

Corollary Let $2 \le a \le d+1$. For all $X \subseteq \mathbb{R}^d$ of size n there exists an a-rainbow set of size $\Omega(n^{1/(2a-1)d})$.

Corollary For all $X \subseteq \mathbb{R}^d$ of size n there exists a 2-rainbow set (dist. distances) of size $\Omega(n^{1/3d})$.

Corollary For all $X \subseteq \mathbb{R}^d$ of size n there is a 3-rainbow set (dist. areas) of size $\Omega(n^{1/5d})$.

Corollary For all $X \subseteq \mathbb{R}^d$ of size n there is a 4-rainbow set (dist. volumes) of size $\Omega(n^{1/7d})$.

Comments on the Proof

1. Proof uses Algebraic Geometry in Proj Space over $\mathbb{C}$.

General Thm

Thm Let $2 \le a \le d+1$. Let $r \in \mathbb{N}$. For all varieties V of dim d and degree r for all sets of n points on V there exists an a-rainbow set of size $\Omega(n^{1/(2a-1)d})$.

Corollary Let $2 \le a \le d+1$. For all $X \subseteq \mathbb{R}^d$ of size n there exists an a-rainbow set of size $\Omega(n^{1/(2a-1)d})$.

Corollary For all $X \subseteq \mathbb{R}^d$ of size n there exists a 2-rainbow set (dist. distances) of size $\Omega(n^{1/3d})$.

Corollary For all $X \subseteq \mathbb{R}^d$ of size n there is a 3-rainbow set (dist. areas) of size $\Omega(n^{1/5d})$.

Corollary For all $X \subseteq \mathbb{R}^d$ of size n there is a 4-rainbow set (dist. volumes) of size $\Omega(n^{1/7d})$.

Comments on the Proof

1. Proof uses Algebraic Geometry in Proj Space over $\mathbb{C}$.
2. Proof uses Maximal subsets in same way as easier proofs.

General Thm

Thm Let $2 \leq a \leq d+1$. Let $r \in \mathbb{N}$. For all varieties V of dim d and degree r for all sets of n points on V there exists an a-rainbow set of size $\Omega(n^{1/(2a-1)d})$.

Corollary Let $2 \leq a \leq d+1$. For all $X \subseteq \mathbb{R}^d$ of size n there exists an a-rainbow set of size $\Omega(n^{1/(2a-1)d})$.

Corollary For all $X \subseteq \mathbb{R}^d$ of size n there exists a 2-rainbow set (dist. distances) of size $\Omega(n^{1/3d})$.

Corollary For all $X \subseteq \mathbb{R}^d$ of size n there is a 3-rainbow set (dist. areas) of size $\Omega(n^{1/5d})$.

Corollary For all $X \subseteq \mathbb{R}^d$ of size n there is a 4-rainbow set (dist. volumes) of size $\Omega(n^{1/7d})$.

Comments on the Proof

1. Proof uses Algebraic Geometry in Proj Space over $\mathbb{C}$.
2. Proof uses Maximal subsets in same way as easier proofs.
3. Proof is by induction on d.

Open Questions

Open Questions

1. Better Particular Results: e.g., want
   for all $X \subseteq \mathbb{R}^2$ *of size* n*, there exists a rainbow set of size* $\Omega(n^{1/2})$*.*

Open Questions

1. Better Particular Results: e.g., want
   for all $X \subseteq \mathbb{R}^2$ *of size* n*, there exists a rainbow set of size* $\Omega(n^{1/2})$*.*
2. General Better Results: e.g., want
   Let $1 \leq a \leq d+1$*. For all* $X \subseteq \mathbb{R}^d$ *of size* n *there exists a rainbow set of size* $\Omega(n^{1/ad})$*.*

Open Questions

1. Better Particular Results: e.g., want
   for all $X \subseteq \mathbb{R}^2$ *of size* n*, there exists a rainbow set of size* $\Omega(n^{1/2})$*.*
2. General Better Results: e.g., want
   Let $1 \leq a \leq d+1$*. For all* $X \subseteq \mathbb{R}^d$ *of size* n *there exists a rainbow set of size* $\Omega(n^{1/ad})$*.*
3. Get easier proofs of general theorem.

Open Questions

1. Better Particular Results: e.g., want *for all $X \subseteq \mathbb{R}^2$ of size n, there exists a rainbow set of size $\Omega(n^{1/2})$.*
2. General Better Results: e.g., want *Let $1 \le a \le d+1$. For all $X \subseteq \mathbb{R}^d$ of size n there exists a rainbow set of size $\Omega(n^{1/ad})$.*
3. Get easier proofs of general theorem.
4. Find any nontrivial limits on what we can do. (Trivial: $n^{1/d}$).

Open Questions

1. Better Particular Results: e.g., want
   for all $X \subseteq \mathbb{R}^2$ *of size* n*, there exists a rainbow set of size* $\Omega(n^{1/2})$*.*
2. General Better Results: e.g., want
   Let $1 \le a \le d+1$*. For all* $X \subseteq \mathbb{R}^d$ *of size* n *there exists a rainbow set of size* $\Omega(n^{1/ad})$*.*
3. Get easier proofs of general theorem.
4. Find any nontrivial limits on what we can do. (Trivial: $n^{1/d}$).
5. Algorithmic aspects.