

BILL, RECORD LECTURE!!!!

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Folkman's Theorem

Exposition by William Gasarch

September 26, 2025

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More generally, we want **all** non-empty sums are the same color.

$b_1 < b_2$ Theorem

Thm $(\forall c)(\exists T = T(c))$ st $\forall \text{COL} : [T] \rightarrow [c] \exists b_1 < b_2$ st
 $\text{COL}(b_2) = \text{COL}(b_1 + b_2)$

Let $T = 3c$ (this is prob not optimal).

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Note $b_1 < b_2$ thm follows from Schur, but we wanted elt proof.

$b_1 < b_2$ Theorem With Divisibility

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 $\text{COL}(b_2) = \text{COL}(b_1 + b_2)$ AND $b_1 < b_2$ AND
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WRITE DOWN WHAT $T(c, d)$ MEANS FOR LATER USE.

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All sums with last term b_1 have same color (trivial).

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Note that these can be different colors.

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Will prove on next slides.

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We later show general case of $b_1 < \dots < b_n$.

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Fix c . U is TBD. Assume there is COL: $[U] \rightarrow [c]$.

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U will be in two blocks, both **very large**.

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Note that $d \leq \frac{W(k, c)}{k}$.

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Don't know d . **Boo!** But know $d \leq \frac{W(k, c)}{k} \leq W(k, c)$. **Yeah!**

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$(\exists a, d)[a, a + d, \dots, a + (k - 1)d \text{ same color}]$ a will be b_3 .

Note that $d \leq \frac{W(k, c)}{k}$.

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Don't know d . **Boo!** But know $d \leq \frac{W(k, c)}{k} \leq W(k, c)$. **Yeah!**

So take Block1 to be $T(c)W(k, c)$.

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Proof of $b_1 < b_2 < b_3$ Theorem (cont)

Recap proof so far

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Upshot Need $b'_2 + b'_1 \leq k$. Take $k = 2T(c) = 6c$.

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$$U(c) = 3cW(6c, c) + W(6c, c) = (3c + 1)W(6c, c).$$

Summarize Proof

$$U = 2W(k, c) \text{ where } k = 2W(dT(c), c).$$

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$b_1 < \dots < b_n$ Theorem

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Will prove on next slides.

Proof of $b_1 < \dots < b_n$ Theorem

Fix c . U is TBD. Assume there COL: $[U] \rightarrow [c]$.

Proof of $b_1 < \dots < b_n$ Theorem

Fix c . U is TBD. Assume there COL: $[U] \rightarrow [c]$.

Let $k = 2U(n-1, c)$. $U(n, c) = 2W(k, c)$. 2 blocks.

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Thm $(\forall n, c)(\exists F = F(n, c))(\forall \text{COL}[F] \rightarrow [c])(\exists x_1, \dots, x_n)$ st
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Proof on Next Slide.

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Call those n x_1, \dots, x_n .

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