

BILL, RECORD LECTURE!!!!

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An Application of Ramsey's Theorem to Logic

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We will assume E is symmetric and not reflexive.

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There is x_1, x_2 st x_1 connects to **every** vertex (except x_2), and x_2 connects to NO other vertex.

For all $n \geq 2$ there is G with n vertex that satisfies this sentence.

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Definition If ϕ is a sentence in the language of graphs then $\text{spec}(\phi)$ is the set of all n such that there is G on n vertices such that $G \models \phi$.

Spectrum: Examples

$$\phi = (\exists x_1, x_2, x_3)[E(x_1, x_2) \wedge E(x_1, x_3)]$$

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 $(\exists G \text{ on } 0 \text{ vertices})[G \models \phi]$? NO.

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$(\forall n \geq 3)(\exists G \text{ on } n \text{ vertices})[G \models \phi]$. YES.

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$(\exists G \text{ on } 3 \text{ vertices})[G \models \phi]$? NO. Discuss.

$\text{spec}(\phi) = \{0, 2, 4, 6, \dots, \}$

Spectrum: Another Example

$(\forall x_1, x_2, x_3)$

[
 $\neg(E(x_1, x_2) \wedge E(x_1, x_3) \wedge E(x_2, x_3))$

\wedge

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Discuss

Spectrum: Another Example

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This is asking for a graph without a 3-clique or 3-ind set.

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$\text{spec}(\phi) = \{0, 1, 2, 3, 4, 5\}$.

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What is spec? Discuss.

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$\text{spec}(\phi) = \{2\}$.

Note how Simple Those Spectrum's Were

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All of these sentence were of the form $(\exists^* \forall^*)$.

$$(\exists x_1) \cdots (\exists x_n)(\forall y_1) \cdots (\forall y_m)[\psi(x_1, \dots, x_n, y_1, y \dots, y_m)]$$

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Or is there a Theorem? Does the proof **Use Ramsey Theory**?

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Proof Use brute force.

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Note For many (ϕ, G) can do much better than brute force.

Main Theorem

Theorem The following function is computable: Given ϕ , an $\exists^*\forall^*$ sentence in the theory of graphs, output $\text{spec}(\phi)$. ($\text{spec}(\phi)$ will be a finite or cofinite set; hence it will have an easy description.)

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We will take

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End of Proof of Claim 1

Claim 2, The Main Claim

$$\phi = (\exists x_1) \cdots (\exists x_n) (\forall y_1) \cdots (\forall y_m) [\psi(x_1, \dots, x_n, y_1, \dots, y_m)]$$

If $(\exists N \geq n + 2^n R(m)) [N \in \text{spec}(\phi)]$ then

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Proof of Claim 2

Since $N \in \text{spec}(\phi)$ there exists $G = (V, E)$, a graph on N vertices such that $G \models \phi$. Let v_1, \dots, v_n be such that:

$$(\forall y_1) \cdots (\forall y_m) [\psi(v_1, \dots, v_n, y_1, \dots, y_m)].$$

(Proof continued on next slide)

Proof of Claim 2 Continued

$$(\forall y_1) \cdots (\forall y_m) [\psi(v_1, \dots, v_n, y_1, \dots, y_m)].$$

Let $X = \{v_1, \dots, v_n\}$ and $U = V - X$.

Proof of Claim 2 Continued

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Let $X = \{v_1, \dots, v_n\}$ and $U = V - X$.

Note that $|U - X| \geq 2^n R(m)$. We use $2^n R(m)$ elements of it which we denote

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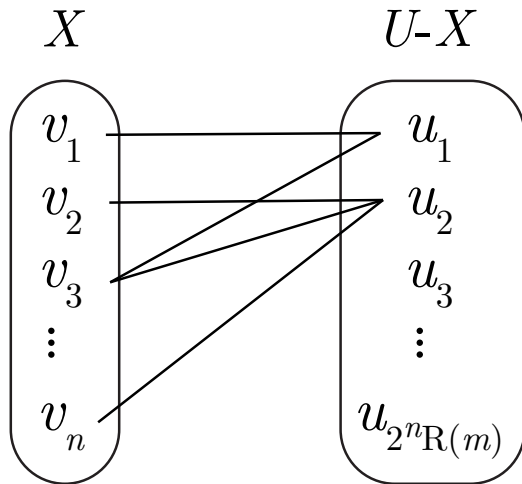
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Picture on next slide.

X and $U - X$



Proof of Claim 2 Cont: Pigeonhole

We define a 2^n -Coloring of U . $u \in U$ maps to (b_1, \dots, b_n) :

$$b_i = \begin{cases} 0 & \text{if } (u, v_i) \notin E \\ 1 & \text{if } (u, v_i) \in E \end{cases} \quad (1)$$

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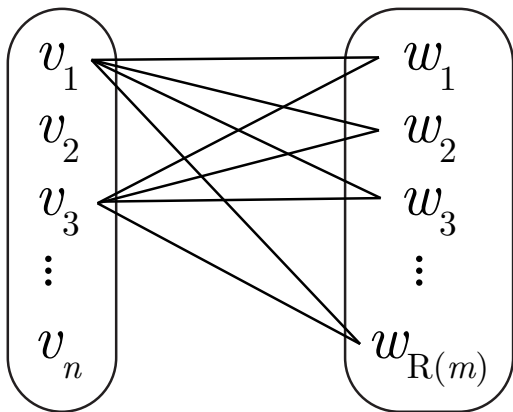
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So they all look the same to U . (Picture on the next slide.)

w_i 's Look the Same to U

X

Pigeonhole



Proof of Claim 2 Cont: Ramsey

Apply Ramsey's Theorem to the graph on

$$\{w_1, \dots, w_{R(m)}\}.$$

to obtain homog set

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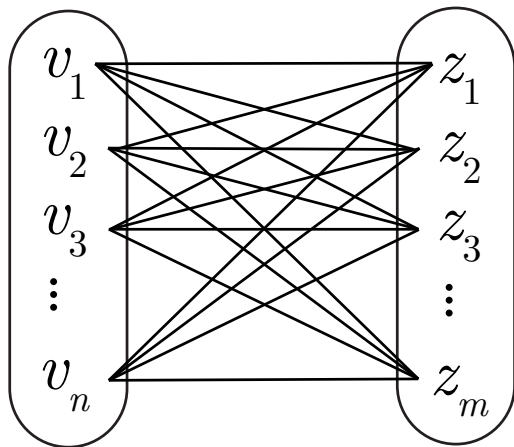
We call the set **Super Homog** since it looks the same to U and to each other.

Picture on the next slide.

The Super Homog Set

X

Homog



Proof of Claim 2 Continued

$$(\exists x_1) \cdots (\exists x_n)(\forall y_1) \cdots (\forall y_m)[\psi(x_1, \dots, x_n, y_1, \dots, y_m)]$$

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For every $p \geq 1$ we form a graph $H_p = (V_p, E_p)$ on $n + m + p$ vertices such that $H_p \models \phi$:

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Informally add $m + p$ vertices that are **just like the z_i 's**.

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Formally Next Slide.

Proof of Claim 2 Continued, Formal $H_p = (V_p, E_p)$

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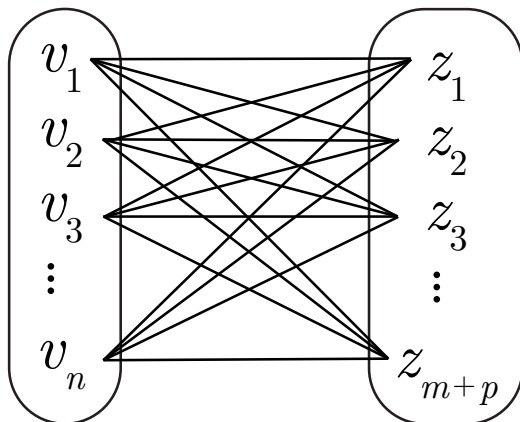
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Can Add Vertices

X

Homog



Claim 3

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$$N_0 = n + 2^n R(m).$$

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Hence, if $N_0 \notin \text{spec}(\phi)$, none of $N_0 + p$ are in $\text{spec}(\phi)$.

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End of Proof of Claim 3

Recap Both Claims

We put a subcase of Claim 2, and Claim 3, next to each other to recap what we know.

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2. Let $N_0 = n + 2^n R(m)$. Determine if $N_0 \in \text{spec}(\phi)$.

2.1 If YES then by Claim 2 $\{n + m, \dots\} \subseteq \text{spec}(\phi)$.

For $0 \leq i \leq n + m - 1$ test if $i \in \text{spec}(\phi)$. You now know $\text{spec}(\phi)$ which is co-finite. Output it.

Algorithm for Finding $\text{spec}(\phi)$

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End of Proof of Main Theorem

Other Sentences. Part I

What other Sentences could we look at?

$\exists^* \forall^*$ sentences with more complicated objects than graphs.

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a -ary Hypergraphs a -ary Hyperedges. c -colored hypergraph. For example every triple is colored R or B or G . Or any finite number of colors.

Key ingredient Ramsey theory on a -hypergraphs.

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Many Predicates We could have $U_1(x), \dots, U_7(x)$ (all 19-valued).
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GUESS is 19^7 .

Key ingredient Ramsey theory on $\leq a$ -hypergraphs since use it on Unary THEN binary, THEN (if more arities then more).

Mason Sentences

$(\exists^* \forall^*)^*$ -sentences, only predicate $E(x, y)$. **Mason** sentences.

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Answer on next slide.

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Known If A is a Union of AP's then $(\exists\phi)[\text{spec}(\phi) = A]$.

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$(\exists^* \forall^*)^*$ -sentences, predicates of arity $\leq a$ -ary. **McKenzie**
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Is spec for McKenzie Sentences decidable? **Vote**.

YES, NO, Unknown to Science. YES.

Known If ϕ is a Mackenzie sentence then $\text{spec}(\phi) \in EXPTIME$.

Also Known If $A \in EXPTIME$ then there exists Mackenzie ϕ such that $\text{spec}(\phi) = A$.

App, “App”, or ““App””

Vote App OR “App” OR ““App””

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““App”” This would be unfair. I reserve the 4-quotes if either NOBODY cares or ONLY I care. (When I prove primes are infinite FROM van Der Waerden’s Theorem, feel free to use 4 quotes. I am not kidding.)

Vote App OR “App” OR ““App””