

BILL, RECORD LECTURE!!!!

BILL RECORD LECTURE!!!

Ramsey's Theorem for Infinite Bipartite Graphs

Exposition by William Gasarch-U of MD

Infinite Ramsey Theory

for $K_{\mathbb{N},\mathbb{N}}$

Ramsey Theory for $\mathbb{N} \times \mathbb{N}$

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Thm $\exists \text{COL}: \mathbb{N} \times \mathbb{N} \rightarrow [c]$ no $c - 1$ -homog $H_1 \times H_2$.

$$c \geq 2$$

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$$\text{COL}(x, y) = \begin{cases} 1 & \text{if } x < y \\ 2 & \text{if } x > y \end{cases} \quad (1)$$

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$$\text{COL}(x, y) = \begin{cases} 1 & \text{if } x < y \\ 2 & \text{if } x > y \end{cases} \quad (1)$$

There is no 1-homog $H_1 \times H_2 \subseteq \mathbb{N} \times \mathbb{N}$. Left to the reader.

$$c \leq 2$$

$\forall \text{ COL: } \mathbb{N} \times \mathbb{N} \rightarrow [1, 000, 000] \exists \mathbf{H}_1 \times \mathbf{H}_2, \text{ 2-homog.}$

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$\forall \text{ COL}: \mathbb{N} \times \mathbb{N} \rightarrow [1, 000, 000] \exists \mathbf{H}_1 \times \mathbf{H}_2, \text{ 2-homog.}$

We do an example. The formal construction is left to the reader.

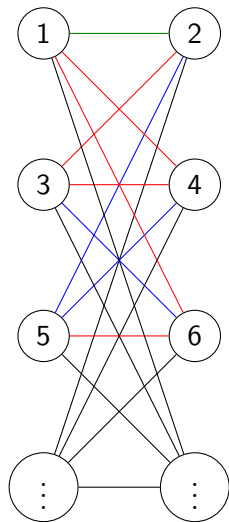
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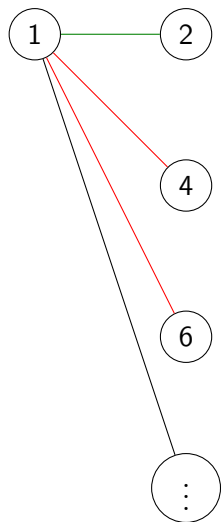
We do an example. The formal construction is left to the reader.

Initially we have $\text{COL}: \mathbb{N} \times \mathbb{N} \rightarrow [1, 000, 000]$.

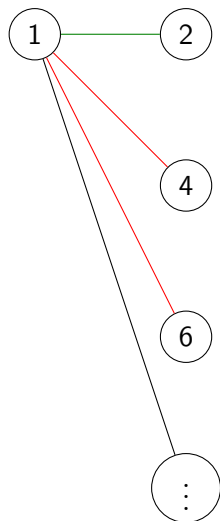
Example of finite coloring of $\mathbb{N} \times \mathbb{N}$



Focus on Vertex 1 On The Left

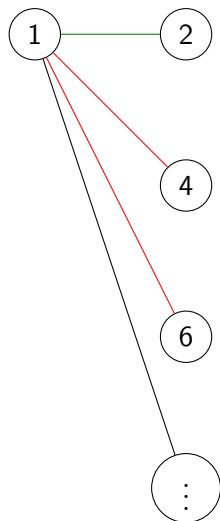


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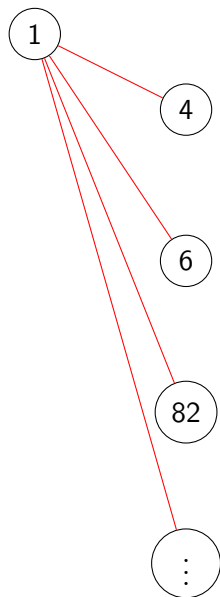
Let c be least color such that $\exists^\infty x, \text{COL}(1, x) = c$. We assume R .

Focus on Vertex 1 On The Left

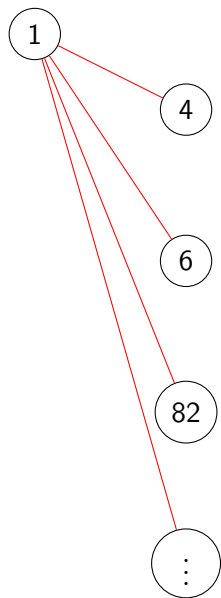


Let c be least color such that $\exists^\infty x, \text{COL}(1, x) = c$. We assume R .
Kill All Those On The Right Who Disagree.

Focus on Vertex 1 On The Left After The Massacre

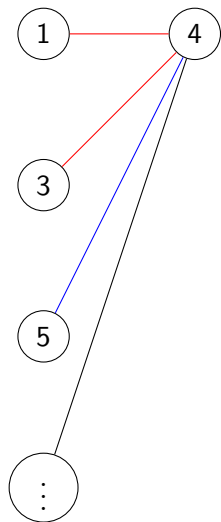


Focus on Vertex 1 On The Left After The Massacre

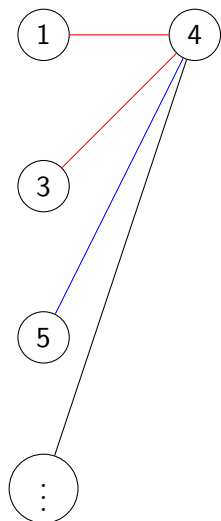


1 is **“immortal”**. We focus on 4.

Focusing on 4 On The Right

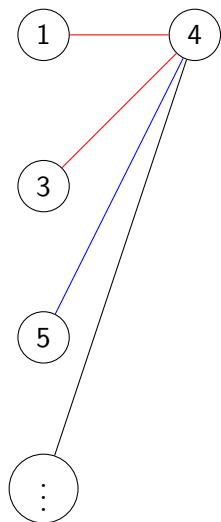


Focusing on 4 On The Right



Let c be least color such that $\exists^\infty x, \text{COL}(x, 4) = c$. We assume B .

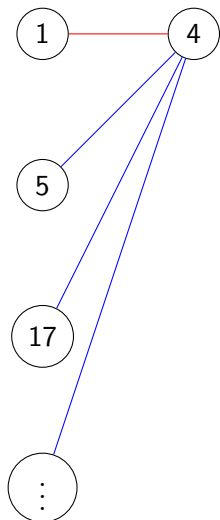
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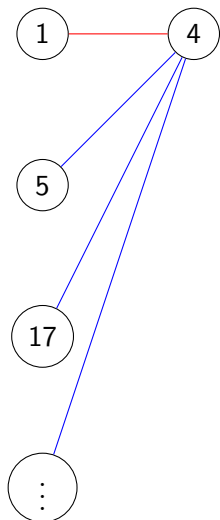
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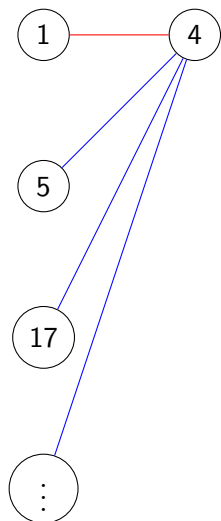


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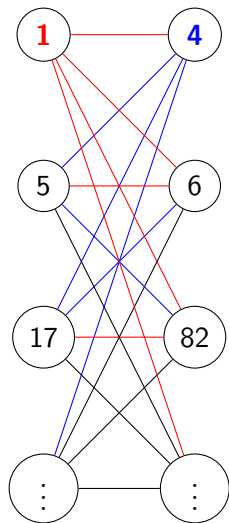
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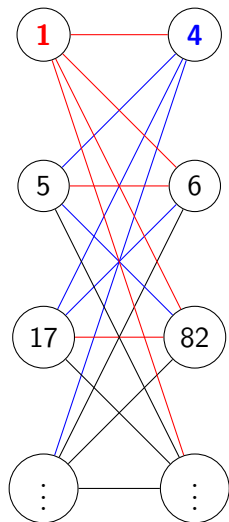
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We Have Processed 1 and 4

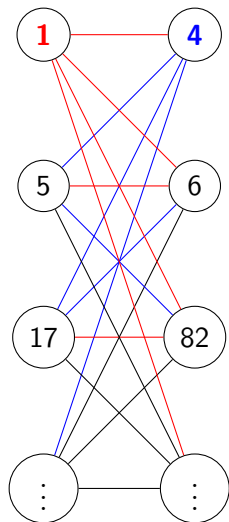


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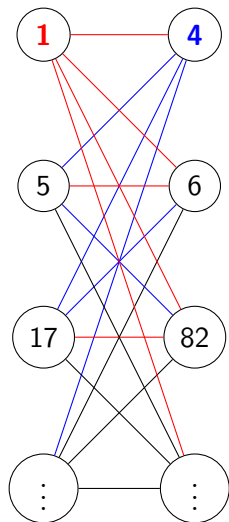
1 is colored *R*. 4 is colored *B*. 1,4 “immortal”

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1 is colored R . 4 is colored B . 1,4 “immortal”
4 is B even though one of the edges out of it is R .

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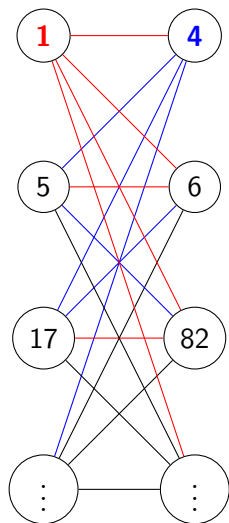


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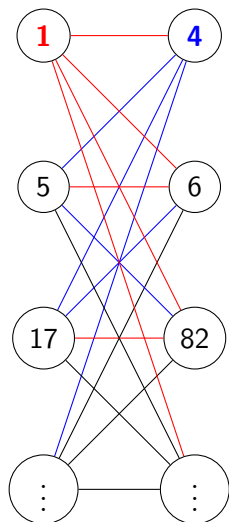
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Key If $x > 4$ then $\text{COL}(x, 4) = B$.

We Have ..., Now What?

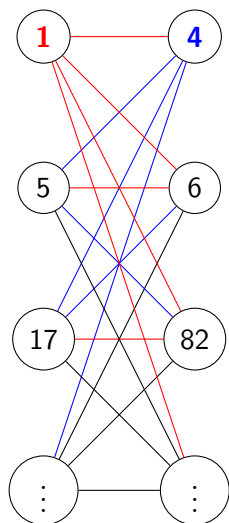


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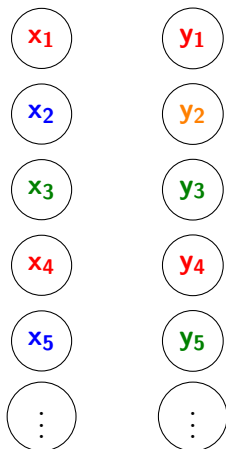
Process 5, Color 5, Process 6, Color 6. Both **“immortal”**

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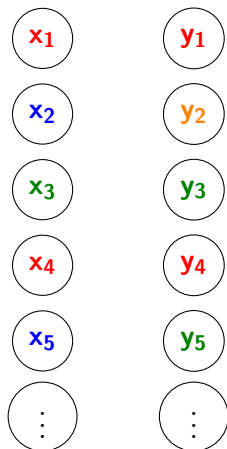


Process 5, Color 5, Process 6, Color 6. Both **“immortal”**
Repeat Process.

So You Thought You Were Immortal. HA!

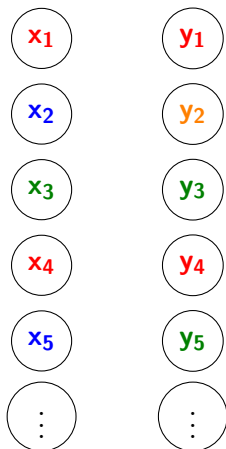


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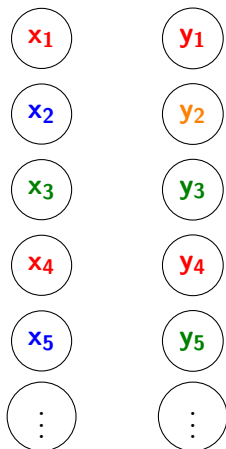
Finite number of colors (though could be large)

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 $(\exists c)(\exists^\infty x)[\text{COL}'(x) = c]$. Assume R .

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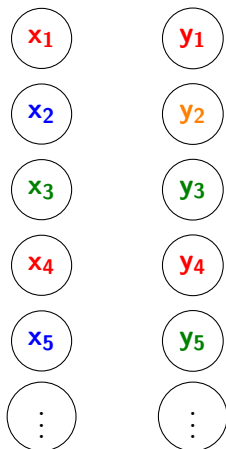


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$(\exists d)(\exists^\infty x)[\text{COL}'(y) = d]$. Assume G .

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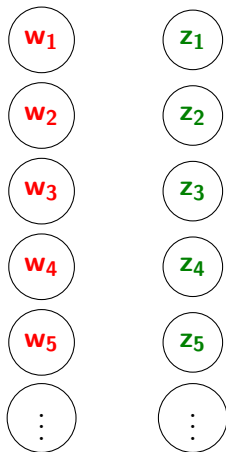
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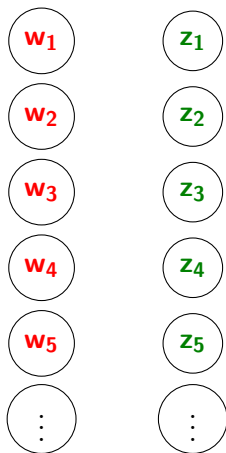
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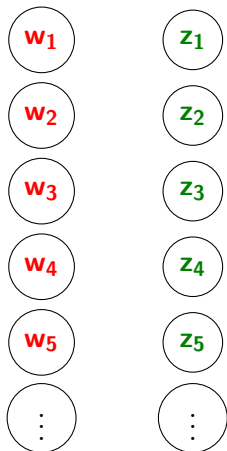


We Are Done!



$$H_1 = \{w_1, w_2, \dots\}$$

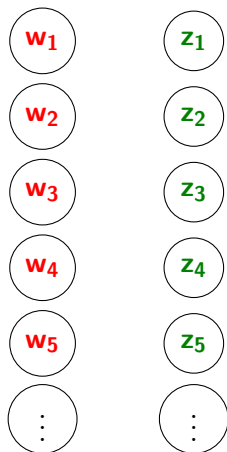
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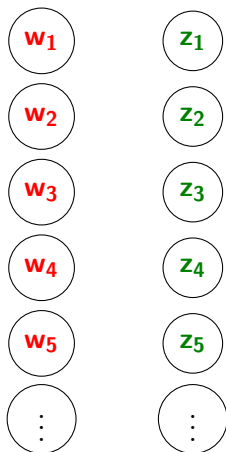
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Began with finite COL: $\mathbb{N} \times \mathbb{N} \rightarrow [1, 000, 000]$.

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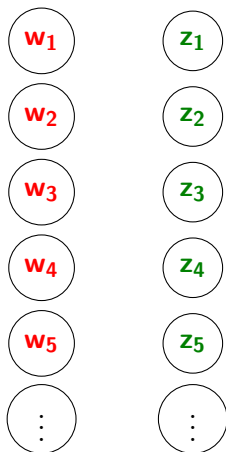


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Restate theorem we just proved on next slide.

Recap

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Thm $\forall \text{COL}: \mathbb{N} \times \mathbb{N} \rightarrow [1, 000, 000] \exists$ 2-homog (H_1, H_2) .

$$\omega + \omega + \omega$$

Theorem for $\omega + \omega + \omega$

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Thm \forall COL: $(\omega + \omega + \omega)_2 \rightarrow [1, 000, 000] \exists$ XXX-homog
 $H \equiv \omega + \omega + \omega$.

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More to come!

Edges Between ω 's

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1-2) View the edges from H_1 to H_2 as a bipartite graph.
Use Bipartite Thm. Colors $c_{1.2.1}$, $c_{1.2.2}$

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Use Bipartite Thm. Colors $c_{2.3.1}$, $c_{2.3.2}$

Colors: c_1 , c_2 , c_3 , $c_{1.2.1}$, $c_{1.2.2}$, $c_{1.3.1}$, $c_{1.3.2}$, $c_{2.3.1}$, $c_{2.3.2}$

9 colors.

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Use Bipartite Thm. Colors $c_{2.3.1}$, $c_{2.3.2}$

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9 colors.

Is this optimal? Yes. Next slide.

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We define COL: $(\omega^{\omega^{\omega}})_2 \rightarrow [9]$.

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- 1) Pairs within the 1st (2nd, 3rd) ω are all colored 1 (2,3)
- 2) Edges between 1st and 2nd omega. Assume i in 1st ω , j in 2nd ω .
COL(i, j) = 4 if $i < j$, 5 if $i > j$.

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4) Edges between 2nd and 3rd omega. Assume i in 2nd ω , j in 3rd ω .

$\text{COL}(i,j) = 8$ if $i < j$, 9 if $i > j$.

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Claim there are no 8-homog sets.

No 8-Homog Sets

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So 9 colors are used.

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Proof of optimality is similar to the $\omega + \omega + \omega$ case.