BILL, RECORD LECTURE!!!!

BILL RECORD LECTURE!!!

Application! Restricting Domains To Stop Being Onto

Exposition by William Gasarch

March 26, 2026

Can Always Find $\mathbb{D}$

Can Always Find $\mathbb{D}$

Thm $\forall f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}\ \exists$ infinite $\mathbb{D} \subseteq \mathbb{Z}$, $f : \mathbb{D} \times \mathbb{D}$ is not onto.

Can Always Find $\mathbb{D}$

Thm $\forall f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}\ \exists$ infinite $\mathbb{D} \subseteq \mathbb{Z}$, $f : \mathbb{D} \times \mathbb{D}$ is not onto.

Given f, we will use Ramsey's Theorem 3 times.

Can Always Find $\mathbb{D}$

Thm $\forall f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}\ \exists$ infinite $\mathbb{D} \subseteq \mathbb{Z}$, $f : \mathbb{D} \times \mathbb{D}$ is not onto.

Given f, we will use Ramsey's Theorem 3 times.

Why 3? We will discuss that later.

First Use of Ramsey

First Use of Ramsey

Define $\mathrm{COL}_1 : \mathbb{Z} \to [4]$ via

First Use of Ramsey

Define $\mathrm{COL}_1 : \mathbb{Z} \to [4]$ via

$$\mathrm{COL}_1(x) = \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right.$$

First Use of Ramsey

Define $\mathrm{COL}_1 : \mathbb{Z} \rightarrow [4]$ via

$$\mathrm{COL}_1(x) = \begin{cases} 0 \text{ if } f(x,x) = 0 \\ \\ \\ \end{cases}$$

First Use of Ramsey

Define $\mathrm{COL}_1 \colon \mathbb{Z} \to [4]$ via

$$\mathrm{COL}_1(x) = \begin{cases} 0 \text{ if } f(x,x) = 0 \\ 1 \text{ if } f(x,x) = 1 \\ \\ \\ \end{cases}$$

First Use of Ramsey

Define $\mathrm{COL}_1 : \mathbb{Z} \to [4]$ via

$$\mathrm{COL}_1(x) = \begin{cases} 0 \text{ if } f(x,x) = 0 \\ 1 \text{ if } f(x,x) = 1 \\ 2 \text{ if } f(x,x) = 2 \\ \end{cases}$$

First Use of Ramsey

Define $\mathrm{COL}_1 \colon \mathbb{Z} \to [4]$ via

$$\mathrm{COL}_1(x) = \begin{cases} 0 \text{ if } f(x,x) = 0 \\ 1 \text{ if } f(x,x) = 1 \\ 2 \text{ if } f(x,x) = 2 \\ \mathbf{R} \text{ if } f(x,x) \notin \{0,1,2\} \end{cases}$$

First Use of Ramsey

Define $\mathrm{COL}_1 : \mathbb{Z} \rightarrow [4]$ via

$$\mathrm{COL}_1(x) = \begin{cases} 0 \text{ if } f(x,x) = 0 \\ 1 \text{ if } f(x,x) = 1 \\ 2 \text{ if } f(x,x) = 2 \\ \mathbf{R} \text{ if } f(x,x) \notin \{0,1,2\} \end{cases}$$

Let H_1 be the infinite homog set.

First Use of Ramsey

Define $\mathrm{COL}_1 : \mathbb{Z} \rightarrow [4]$ via

$$\mathrm{COL}_1(x) = \begin{cases} 0 \text{ if } f(x,x) = 0 \\ 1 \text{ if } f(x,x) = 1 \\ 2 \text{ if } f(x,x) = 2 \\ \mathbf{R} \text{ if } f(x,x) \notin \{0,1,2\} \end{cases}$$

Let H_1 be the infinite homog set.
If $\mathrm{COL}_1(H_1) = 0$ then $(\forall x \in H_1)[f(x,x) = 0]$.

First Use of Ramsey

Define $\mathrm{COL}_1 : \mathbb{Z} \rightarrow [4]$ via

$$\mathrm{COL}_1(x) = \begin{cases} 0 \text{ if } f(x,x) = 0 \\ 1 \text{ if } f(x,x) = 1 \\ 2 \text{ if } f(x,x) = 2 \\ \mathbf{R} \text{ if } f(x,x) \notin \{0,1,2\} \end{cases}$$

Let H_1 be the infinite homog set.
If $\mathrm{COL}_1(H_1) = 0$ then $(\forall x \in H_1)[f(x,x) = 0]$.
If $\mathrm{COL}_1(H_1) = 1$ then $(\forall x \in H_1)[f(x,x) = 1]$.

First Use of Ramsey

Define $\mathrm{COL}_1 : \mathbb{Z} \rightarrow [4]$ via

$$\mathrm{COL}_1(x) = \begin{cases} 0 \text{ if } f(x,x) = 0 \\ 1 \text{ if } f(x,x) = 1 \\ 2 \text{ if } f(x,x) = 2 \\ \mathbf{R} \text{ if } f(x,x) \notin \{0,1,2\} \end{cases}$$

Let H_1 be the infinite homog set.
If $\mathrm{COL}_1(H_1) = 0$ then $(\forall x \in H_1)[f(x,x) = 0]$.
If $\mathrm{COL}_1(H_1) = 1$ then $(\forall x \in H_1)[f(x,x) = 1]$.
If $\mathrm{COL}_1(H_1) = 2$ then $(\forall x \in H_1)[f(x,x) = 2]$.

First Use of Ramsey

Define $\mathrm{COL}_1 : \mathbb{Z} \rightarrow [4]$ via

$$\mathrm{COL}_1(x) = \begin{cases} 0 \text{ if } f(x,x) = 0 \\ 1 \text{ if } f(x,x) = 1 \\ 2 \text{ if } f(x,x) = 2 \\ \textbf{R} \text{ if } f(x,x) \notin \{0,1,2\} \end{cases}$$

Let H_1 be the infinite homog set.
If $\mathrm{COL}_1(H_1) = 0$ then $(\forall x \in H_1)[f(x,x) = 0]$.
If $\mathrm{COL}_1(H_1) = 1$ then $(\forall x \in H_1)[f(x,x) = 1]$.
If $\mathrm{COL}_1(H_1) = 2$ then $(\forall x \in H_1)[f(x,x) = 2]$.
If $\mathrm{COL}_1(H_1) = \textbf{R}$ then $(\forall x \in H_1)[f(x,x) \notin \{0,1,2\}]$.

First Use of Ramsey

Define $\mathrm{COL}_1 : \mathbb{Z} \rightarrow [4]$ via

$$\mathrm{COL}_1(x) = \begin{cases} 0 \text{ if } f(x,x) = 0 \\ 1 \text{ if } f(x,x) = 1 \\ 2 \text{ if } f(x,x) = 2 \\ \mathbf{R} \text{ if } f(x,x) \notin \{0,1,2\} \end{cases}$$

Let H_1 be the infinite homog set.
If $\mathrm{COL}_1(H_1) = 0$ then $(\forall x \in H_1)[f(x,x) = 0]$.
If $\mathrm{COL}_1(H_1) = 1$ then $(\forall x \in H_1)[f(x,x) = 1]$.
If $\mathrm{COL}_1(H_1) = 2$ then $(\forall x \in H_1)[f(x,x) = 2]$.
If $\mathrm{COL}_1(H_1) = \mathbf{R}$ then $(\forall x \in H_1)[f(x,x) \notin \{0,1,2\}]$.
Not done- that's just $f(x,x)$.

First Use of Ramsey

Define $\mathrm{COL}_1 : \mathbb{Z} \rightarrow [4]$ via

$$\mathrm{COL}_1(x) = \begin{cases} 0 \text{ if } f(x,x) = 0 \\ 1 \text{ if } f(x,x) = 1 \\ 2 \text{ if } f(x,x) = 2 \\ \mathbf{R} \text{ if } f(x,x) \notin \{0,1,2\} \end{cases}$$

Let H_1 be the infinite homog set.
If $\mathrm{COL}_1(H_1) = 0$ then $(\forall x \in H_1)[f(x,x) = 0]$.
If $\mathrm{COL}_1(H_1) = 1$ then $(\forall x \in H_1)[f(x,x) = 1]$.
If $\mathrm{COL}_1(H_1) = 2$ then $(\forall x \in H_1)[f(x,x) = 2]$.
If $\mathrm{COL}_1(H_1) = \mathbf{R}$ then $(\forall x \in H_1)[f(x,x) \notin \{0,1,2\}]$.
Not done- that's just $f(x,x)$.
Will now look at f restricted to $(x,y) \in H_1 \times H_1$ with $x < y$.

Second Use of Ramsey

Second Use of Ramsey

Define $\mathrm{COL}_2 \colon \binom{H_1}{2} \to [4]$

Second Use of Ramsey

Define $\mathrm{COL}_2 \colon \binom{H_1}{2} \to [4]$
Recall that the coloring is on **unordered pairs**

Second Use of Ramsey

Define $\mathrm{COL}_2 \colon \binom{H_1}{2} \to [4]$
Recall that the coloring is on **unordered pairs**
COL_2 takes input $\{x, y\}$ and we can assume $x < y$.

Second Use of Ramsey

Define $\mathrm{COL}_2 \colon \binom{H_1}{2} \to [4]$
Recall that the coloring is on **unordered pairs**
COL_2 takes input $\{x, y\}$ and we can assume $x < y$.

$$\mathrm{COL}_2(x, y) = \begin{cases} \\ \\ \end{cases}$$

Second Use of Ramsey

Define $\mathrm{COL}_2\colon \binom{H_1}{2} \to [4]$
Recall that the coloring is on **unordered pairs**
COL_2 takes input $\{x, y\}$ and we can assume $x < y$.

$$\mathrm{COL}_2(x, y) = \begin{cases} 0 \text{ if } f(x, y) = 0 \\ \\ \\ \end{cases}$$

Second Use of Ramsey

Define $\mathrm{COL}_2 \colon \binom{H_1}{2} \rightarrow [4]$
Recall that the coloring is on **unordered pairs**
COL_2 takes input $\{x, y\}$ and we can assume $x < y$.

$$\mathrm{COL}_2(x, y) = \begin{cases} 0 \text{ if } f(x, y) = 0 \\ 1 \text{ if } f(x, y) = 1 \\ \\ \\ \end{cases}$$

Second Use of Ramsey

Define $\mathrm{COL}_2 \colon \binom{H_1}{2} \to [4]$
Recall that the coloring is on **unordered pairs**
COL_2 takes input $\{x, y\}$ and we can assume $x < y$.

$$\mathrm{COL}_2(x, y) = \begin{cases} 0 \text{ if } f(x, y) = 0 \\ 1 \text{ if } f(x, y) = 1 \\ 2 \text{ if } f(x, y) = 2 \\ \end{cases}$$

Second Use of Ramsey

Define $\mathrm{COL}_2 \colon \binom{H_1}{2} \to [4]$
Recall that the coloring is on **unordered pairs**
COL_2 takes input $\{x, y\}$ and we can assume $x < y$.

$$\mathrm{COL}_2(x,y) = \begin{cases} 0 \text{ if } f(x,y) = 0 \\ 1 \text{ if } f(x,y) = 1 \\ 2 \text{ if } f(x,y) = 2 \\ \mathbf{R} \text{ if } f(x,y) \notin \{0,1,2\} \end{cases} \tag{1}$$

Second Use of Ramsey

Define $\mathrm{COL}_2 \colon \binom{H_1}{2} \to [4]$
Recall that the coloring is on **unordered pairs**
COL_2 takes input $\{x, y\}$ and we can assume $x < y$.

$$\mathrm{COL}_2(x, y) = \begin{cases} 0 \text{ if } f(x, y) = 0 \\ 1 \text{ if } f(x, y) = 1 \\ 2 \text{ if } f(x, y) = 2 \\ \mathbf{R} \text{ if } f(x, y) \notin \{0, 1, 2\} \end{cases} \tag{1}$$

Let $H_2 \subseteq H_1$ be the infinite homog set.

Second Use of Ramsey

Define $\mathrm{COL}_2 \colon \binom{H_1}{2} \to [4]$
Recall that the coloring is on **unordered pairs**
COL_2 takes input $\{x, y\}$ and we can assume $x < y$.

$$\mathrm{COL}_2(x,y) = \begin{cases} 0 \text{ if } f(x,y) = 0 \\ 1 \text{ if } f(x,y) = 1 \\ 2 \text{ if } f(x,y) = 2 \\ \mathbf{R} \text{ if } f(x,y) \notin \{0,1,2\} \end{cases} \quad (1)$$

Let $H_2 \subseteq H_1$ be the infinite homog set.
If $\mathrm{COL}_2(H_2) = 0$ then $(\forall x, y \in H_2, x < y)[f(x,y) = 0]$.

Second Use of Ramsey

Define $\mathrm{COL}_2\colon \binom{H_1}{2} \to [4]$
Recall that the coloring is on **unordered pairs**
COL_2 takes input $\{x, y\}$ and we can assume $x < y$.

$$\mathrm{COL}_2(x,y) = \begin{cases} 0 \text{ if } f(x,y) = 0 \\ 1 \text{ if } f(x,y) = 1 \\ 2 \text{ if } f(x,y) = 2 \\ \mathbf{R} \text{ if } f(x,y) \notin \{0,1,2\} \end{cases} \tag{1}$$

Let $H_2 \subseteq H_1$ be the infinite homog set.
If $\mathrm{COL}_2(H_2) = 0$ then $(\forall x, y \in H_2, x < y)[f(x,y) = 0]$.
If $\mathrm{COL}_2(H_2) = 1$ then $(\forall x, y \in H_2, x < y)[f(x,y) = 1]$.

Second Use of Ramsey

Define $\mathrm{COL}_2 \colon \binom{H_1}{2} \to [4]$
Recall that the coloring is on **unordered pairs**
COL_2 takes input $\{x, y\}$ and we can assume $x < y$.

$$\mathrm{COL}_2(x,y) = \begin{cases} 0 \text{ if } f(x,y) = 0 \\ 1 \text{ if } f(x,y) = 1 \\ 2 \text{ if } f(x,y) = 2 \\ \mathbf{R} \text{ if } f(x,y) \notin \{0,1,2\} \end{cases} \tag{1}$$

Let $H_2 \subseteq H_1$ be the infinite homog set.
If $\mathrm{COL}_2(H_2) = 0$ then $(\forall x, y \in H_2, x < y)[f(x,y) = 0]$.
If $\mathrm{COL}_2(H_2) = 1$ then $(\forall x, y \in H_2, x < y)[f(x,y) = 1]$.
If $\mathrm{COL}_2(H_2) = 2$ then $(\forall x, y \in H_2, x < y)[f(x,y) = 2]$.

Second Use of Ramsey

Define $\mathrm{COL}_2\colon \binom{H_1}{2} \rightarrow [4]$
Recall that the coloring is on **unordered pairs**
COL_2 takes input $\{x, y\}$ and we can assume $x < y$.

$$\mathrm{COL}_2(x,y) = \begin{cases} 0 \text{ if } f(x,y) = 0 \\ 1 \text{ if } f(x,y) = 1 \\ 2 \text{ if } f(x,y) = 2 \\ \mathbf{R} \text{ if } f(x,y) \notin \{0,1,2\} \end{cases} \quad (1)$$

Let $H_2 \subseteq H_1$ be the infinite homog set.
If $\mathrm{COL}_2(H_2) = 0$ then $(\forall x, y \in H_2, x < y)[f(x,y) = 0]$.
If $\mathrm{COL}_2(H_2) = 1$ then $(\forall x, y \in H_2, x < y)[f(x,y) = 1]$.
If $\mathrm{COL}_2(H_2) = 2$ then $(\forall x, y \in H_2, x < y)[f(x,y) = 2]$.
If $\mathrm{COL}_2(H_2) = \mathbf{R}$ then $(\forall x, y \in H_1, x < y)[f(x,y) \notin \{0,1,2\}]$.

Second Use of Ramsey

Define $\mathrm{COL}_2 \colon \binom{H_1}{2} \rightarrow [4]$
Recall that the coloring is on **unordered pairs**
COL_2 takes input $\{x, y\}$ and we can assume $x < y$.

$$\mathrm{COL}_2(x,y) = \begin{cases} 0 \text{ if } f(x,y) = 0 \\ 1 \text{ if } f(x,y) = 1 \\ 2 \text{ if } f(x,y) = 2 \\ \mathbf{R} \text{ if } f(x,y) \notin \{0,1,2\} \end{cases} \tag{1}$$

Let $H_2 \subseteq H_1$ be the infinite homog set.
If $\mathrm{COL}_2(H_2) = 0$ then $(\forall x, y \in H_2, x < y)[f(x,y) = 0]$.
If $\mathrm{COL}_2(H_2) = 1$ then $(\forall x, y \in H_2, x < y)[f(x,y) = 1]$.
If $\mathrm{COL}_2(H_2) = 2$ then $(\forall x, y \in H_2, x < y)[f(x,y) = 2]$.
If $\mathrm{COL}_2(H_2) = \mathbf{R}$ then $(\forall x, y \in H_1, x < y)[f(x,y) \notin \{0,1,2\}]$.
Not done yet. On H_2 we control $f(x,x)$, $f(x,y)$ with $x < y$.

Second Use of Ramsey

Define $\mathrm{COL}_2 \colon \binom{H_1}{2} \rightarrow [4]$
Recall that the coloring is on **unordered pairs**
COL_2 takes input $\{x, y\}$ and we can assume $x < y$.

$$\mathrm{COL}_2(x,y) = \begin{cases} 0 \text{ if } f(x,y) = 0 \\ 1 \text{ if } f(x,y) = 1 \\ 2 \text{ if } f(x,y) = 2 \\ \mathbf{R} \text{ if } f(x,y) \notin \{0,1,2\} \end{cases} \quad (1)$$

Let $H_2 \subseteq H_1$ be the infinite homog set.
If $\mathrm{COL}_2(H_2) = 0$ then $(\forall x, y \in H_2, x < y)[f(x,y) = 0]$.
If $\mathrm{COL}_2(H_2) = 1$ then $(\forall x, y \in H_2, x < y)[f(x,y) = 1]$.
If $\mathrm{COL}_2(H_2) = 2$ then $(\forall x, y \in H_2, x < y)[f(x,y) = 2]$.
If $\mathrm{COL}_2(H_2) = \mathbf{R}$ then $(\forall x, y \in H_1, x < y)[f(x,y) \notin \{0,1,2\}]$.
Not done yet. On H_2 we control $f(x,x)$, $f(x,y)$ with $x < y$.
Now look at f restricted to $(x,y) \in H_1 \times H_1$ with $x > y$.

Third Use of Ramsey

Third Use of Ramsey

Define $\mathrm{COL}_3 \colon \binom{H_2}{2} \to [4]$

Third Use of Ramsey

Define $\mathrm{COL}_3 \colon \binom{H_2}{2} \to [4]$
Recall that the coloring is on **unordered pairs**

Third Use of Ramsey

Define $\mathrm{COL}_3 \colon \binom{H_2}{2} \to [4]$
Recall that the coloring is on **unordered pairs**
COL_3 takes input $\{x, y\}$ and we can assume $x > y$.

Third Use of Ramsey

Define $\mathrm{COL}_3 \colon \binom{H_2}{2} \rightarrow [4]$
Recall that the coloring is on **unordered pairs**
COL_3 takes input $\{x, y\}$ and we can assume $x > y$.

$$\mathrm{COL}_3(x, y) = \begin{cases} \end{cases}$$

Third Use of Ramsey

Define $\mathrm{COL}_3 \colon \binom{H_2}{2} \to [4]$
Recall that the coloring is on **unordered pairs**
COL_3 takes input $\{x, y\}$ and we can assume $x > y$.

$$\mathrm{COL}_3(x, y) = \begin{cases} 0 \text{ if } f(x, y) = 0 \\ \\ \\ \end{cases}$$

Third Use of Ramsey

Define $\mathrm{COL}_3 \colon \binom{H_2}{2} \to [4]$
Recall that the coloring is on **unordered pairs**
COL_3 takes input $\{x, y\}$ and we can assume $x > y$.

$$\mathrm{COL}_3(x, y) = \begin{cases} 0 \text{ if } f(x, y) = 0 \\ 1 \text{ if } f(x, y) = 1 \\ \\ \\ \end{cases}$$

Third Use of Ramsey

Define $\mathrm{COL}_3 \colon \binom{H_2}{2} \rightarrow [4]$
Recall that the coloring is on **unordered pairs**
COL_3 takes input $\{x, y\}$ and we can assume $x > y$.

$$\mathrm{COL}_3(x,y) = \begin{cases} 0 \text{ if } f(x,y) = 0 \\ 1 \text{ if } f(x,y) = 1 \\ 2 \text{ if } f(x,y) = 2 \\ \end{cases}$$

Third Use of Ramsey

Define $\mathrm{COL}_3 \colon \binom{H_2}{2} \to [4]$
Recall that the coloring is on **unordered pairs**
COL_3 takes input $\{x, y\}$ and we can assume $x > y$.

$$\mathrm{COL}_3(x, y) = \begin{cases} 0 \text{ if } f(x, y) = 0 \\ 1 \text{ if } f(x, y) = 1 \\ 2 \text{ if } f(x, y) = 2 \\ \mathbf{R} \text{ if } f(x, y) \notin \{0, 1, 2\} \end{cases} \tag{2}$$

Third Use of Ramsey

Define $\mathrm{COL}_3 \colon \binom{H_2}{2} \rightarrow [4]$
Recall that the coloring is on **unordered pairs**
COL_3 takes input $\{x, y\}$ and we can assume $x > y$.

$$\mathrm{COL}_3(x, y) = \begin{cases} 0 \text{ if } f(x, y) = 0 \\ 1 \text{ if } f(x, y) = 1 \\ 2 \text{ if } f(x, y) = 2 \\ \mathbf{R} \text{ if } f(x, y) \notin \{0, 1, 2\} \end{cases} \tag{2}$$

Let $H_3 \subseteq H_2 \subseteq H_1$ be the infinite homog set.

Third Use of Ramsey

Define $\mathrm{COL}_3 \colon \binom{H_2}{2} \rightarrow [4]$
Recall that the coloring is on **unordered pairs**
COL_3 takes input $\{x, y\}$ and we can assume $x > y$.

$$\mathrm{COL}_3(x, y) = \begin{cases} 0 \text{ if } f(x, y) = 0 \\ 1 \text{ if } f(x, y) = 1 \\ 2 \text{ if } f(x, y) = 2 \\ \mathbf{R} \text{ if } f(x, y) \notin \{0, 1, 2\} \end{cases} \tag{2}$$

Let $H_3 \subseteq H_2 \subseteq H_1$ be the infinite homog set.
If $\mathrm{COL}_3(H_3) = 0$ then $(\forall x, y \in H_3, x > y)[f(x, y) = 0]$.

Third Use of Ramsey

Define $\mathrm{COL}_3 \colon \binom{H_2}{2} \rightarrow [4]$
Recall that the coloring is on **unordered pairs**
COL_3 takes input $\{x, y\}$ and we can assume $x > y$.

$$\mathrm{COL}_3(x,y) = \begin{cases} 0 \text{ if } f(x,y) = 0 \\ 1 \text{ if } f(x,y) = 1 \\ 2 \text{ if } f(x,y) = 2 \\ \mathbf{R} \text{ if } f(x,y) \notin \{0,1,2\} \end{cases} \tag{2}$$

Let $H_3 \subseteq H_2 \subseteq H_1$ be the infinite homog set.
If $\mathrm{COL}_3(H_3) = 0$ then $(\forall x, y \in H_3, x > y)[f(x,y) = 0]$.
If $\mathrm{COL}_3(H_3) = 1$ then $(\forall x, y \in H_3, x > y)[f(x,y) = 1]$.

Third Use of Ramsey

Define $\mathrm{COL}_3 \colon \binom{H_2}{2} \to [4]$
Recall that the coloring is on **unordered pairs**
COL_3 takes input $\{x, y\}$ and we can assume $x > y$.

$$\mathrm{COL}_3(x,y) = \begin{cases} 0 \text{ if } f(x,y) = 0 \\ 1 \text{ if } f(x,y) = 1 \\ 2 \text{ if } f(x,y) = 2 \\ \mathbf{R} \text{ if } f(x,y) \notin \{0,1,2\} \end{cases} \tag{2}$$

Let $H_3 \subseteq H_2 \subseteq H_1$ be the infinite homog set.
If $\mathrm{COL}_3(H_3) = 0$ then $(\forall x, y \in H_3, x > y)[f(x,y) = 0]$.
If $\mathrm{COL}_3(H_3) = 1$ then $(\forall x, y \in H_3, x > y)[f(x,y) = 1]$.
If $\mathrm{COL}_3(H_3) = 2$ then $(\forall x, y \in H_3, x > y)[f(x,y) = 2]$.

Third Use of Ramsey

Define $\mathrm{COL}_3 \colon \binom{H_2}{2} \rightarrow [4]$
Recall that the coloring is on **unordered pairs**
COL_3 takes input $\{x, y\}$ and we can assume $x > y$.

$$\mathrm{COL}_3(x,y) = \begin{cases} 0 \text{ if } f(x,y) = 0 \\ 1 \text{ if } f(x,y) = 1 \\ 2 \text{ if } f(x,y) = 2 \\ \mathbf{R} \text{ if } f(x,y) \notin \{0,1,2\} \end{cases} \tag{2}$$

Let $H_3 \subseteq H_2 \subseteq H_1$ be the infinite homog set.
If $\mathrm{COL}_3(H_3) = 0$ then $(\forall x, y \in H_3, x > y)[f(x,y) = 0]$.
If $\mathrm{COL}_3(H_3) = 1$ then $(\forall x, y \in H_3, x > y)[f(x,y) = 1]$.
If $\mathrm{COL}_3(H_3) = 2$ then $(\forall x, y \in H_3, x > y)[f(x,y) = 2]$.
If $\mathrm{COL}_3(H_3) = \mathbf{R}$ then $(\forall x, y \in H_1, x > y)[f(x,y) \notin \{0,1,2\}]$.

Third Use of Ramsey

Define $\mathrm{COL}_3 \colon \binom{H_2}{2} \rightarrow [4]$
Recall that the coloring is on **unordered pairs**
COL_3 takes input $\{x, y\}$ and we can assume $x > y$.

$$\mathrm{COL}_3(x,y) = \begin{cases} 0 \text{ if } f(x,y) = 0 \\ 1 \text{ if } f(x,y) = 1 \\ 2 \text{ if } f(x,y) = 2 \\ \mathbf{R} \text{ if } f(x,y) \notin \{0,1,2\} \end{cases} \tag{2}$$

Let $H_3 \subseteq H_2 \subseteq H_1$ be the infinite homog set.
If $\mathrm{COL}_3(H_3) = 0$ then $(\forall x, y \in H_3, x > y)[f(x,y) = 0]$.
If $\mathrm{COL}_3(H_3) = 1$ then $(\forall x, y \in H_3, x > y)[f(x,y) = 1]$.
If $\mathrm{COL}_3(H_3) = 2$ then $(\forall x, y \in H_3, x > y)[f(x,y) = 2]$.
If $\mathrm{COL}_3(H_3) = \mathbf{R}$ then $(\forall x, y \in H_1, x > y)[f(x,y) \notin \{0,1,2\}]$.
We show that f on $H_3 \times H_3$ is not onto.

f Restricted to $H_3 \times H_3$ is Not Onto

f Restricted to $H_3 \times H_3$ is Not Onto

All pairs are from $H_3 \times H_3$.

f Restricted to $H_3 \times H_3$ is Not Onto

All pairs are from $H_3 \times H_3$.

$f(x,x)$ is always 0, always 1, always 2, or always $\notin \{0,1,2\}$.

f Restricted to $H_3 \times H_3$ is Not Onto

All pairs are from $H_3 \times H_3$.

$f(x,x)$ is always 0, always 1, always 2, or always $\notin \{0,1,2\}$.

$f(x,y)$ with $x < y$ is always 0, always 1, always 2, or always $\notin \{0,1,2\}$.

f Restricted to $H_3 \times H_3$ is Not Onto

All pairs are from $H_3 \times H_3$.

$f(x,x)$ is always 0, always 1, always 2, or always $\notin \{0,1,2\}$.

$f(x,y)$ with $x < y$ is always 0, always 1, always 2, or always $\notin \{0,1,2\}$.

$f(x,y)$ with $x > y$ is always 0, always 1, always 2, or always $\notin \{0,1,2\}$.

f Restricted to $H_3 \times H_3$ is Not Onto

All pairs are from $H_3 \times H_3$.

$f(x, x)$ is always 0, always 1, always 2, or always $\notin \{0, 1, 2\}$.

$f(x, y)$ with $x < y$ is always 0, always 1, always 2, or always $\notin \{0, 1, 2\}$.

$f(x, y)$ with $x > y$ is always 0, always 1, always 2, or always $\notin \{0, 1, 2\}$.

if color 0 then 0 not in the image, so NOT onto.

f Restricted to $H_3 \times H_3$ is Not Onto

All pairs are from $H_3 \times H_3$.

$f(x,x)$ is always 0, always 1, always 2, or always $\notin \{0,1,2\}$.

$f(x,y)$ with $x < y$ is always 0, always 1, always 2, or always $\notin \{0,1,2\}$.

$f(x,y)$ with $x > y$ is always 0, always 1, always 2, or always $\notin \{0,1,2\}$.

if color 0 then 0 not in the image, so NOT onto.

if color 1 then 1 not in the image, so NOT onto.

f Restricted to $H_3 \times H_3$ is Not Onto

All pairs are from $H_3 \times H_3$.

$f(x,x)$ is always 0, always 1, always 2, or always $\notin \{0,1,2\}$.

$f(x,y)$ with $x < y$ is always 0, always 1, always 2, or always $\notin \{0,1,2\}$.

$f(x,y)$ with $x > y$ is always 0, always 1, always 2, or always $\notin \{0,1,2\}$.

if color 0 then 0 not in the image, so NOT onto.

if color 1 then 1 not in the image, so NOT onto.

if color 2 then 2 not in the image, so NOT onto.

f Restricted to $H_3 \times H_3$ is Not Onto

All pairs are from $H_3 \times H_3$.

$f(x,x)$ is always 0, always 1, always 2, or always $\notin \{0,1,2\}$.

$f(x,y)$ with $x < y$ is always 0, always 1, always 2, or always $\notin \{0,1,2\}$.

$f(x,y)$ with $x > y$ is always 0, always 1, always 2, or always $\notin \{0,1,2\}$.

if color 0 then 0 not in the image, so NOT onto.

if color 1 then 1 not in the image, so NOT onto.

if color 2 then 2 not in the image, so NOT onto.

if color **R** then image does not have $\{0,1,2\}$ in it, so NOT onto.

f Why [4]?

f Why [4]?

Why four colors?

f Why $[4]$?

Why four colors?

f can have three kinds of input:

f Why $[4]$?

Why four colors?

f can have three kinds of input:

(x, y) where $x = y$. Color of H_1 controls this.

f Why $[4]$?

Why four colors?

f can have three kinds of input:

(x, y) where $x = y$. Color of H_1 controls this.

(x, y) where $x < y$. Color of H_2 controls this.

f Why $[4]$?

Why four colors?

f can have three kinds of input:

(x, y) where $x = y$. Color of H_1 controls this.

(x, y) where $x < y$. Color of H_2 controls this.

(x, y) where $x > y$. Color of H_3 controls this.

f Why $[4]$?

Why four colors?

f can have three kinds of input:

(x, y) where $x = y$. Color of H_1 controls this.

(x, y) where $x < y$. Color of H_2 controls this.

(x, y) where $x > y$. Color of H_3 controls this.

4 is 1 more than the number possible types of inputs.

f Why $[4]$?

Why four colors?

f can have three kinds of input:

(x, y) where $x = y$. Color of H_1 controls this.

(x, y) where $x < y$. Color of H_2 controls this.

(x, y) where $x > y$. Color of H_3 controls this.

4 is 1 more than the number possible types of inputs.

So some color is not used.

Examples

Example

Examples

Example

H_1 is colored 0. So $f(x, x)$ is always 0. (x, y) where $x = y$.

Examples

Example

H_1 is colored 0. So $f(x, x)$ is always 0. (x, y) where $x = y$.

H_2 is colored 1. So $f(x, y)$ with $x < y$ is always 1.

Examples

Example

H_1 is colored 0. So $f(x,x)$ is always 0. (x,y) where $x=y$.

H_2 is colored 1. So $f(x,y)$ with $x<y$ is always 1.

H_3 is colored **R**. So $f(x,y)$ with $x>y$ is never in $\{0,1,2\}$

Examples

Example

H_1 is colored 0. So $f(x,x)$ is always 0. (x,y) where $x = y$.

H_2 is colored 1. So $f(x,y)$ with $x < y$ is always 1.

H_3 is colored **R**. So $f(x,y)$ with $x > y$ is never in $\{0,1,2\}$

If f is restricted to $D \times D$ then 2 is not hit. So not onto.

Examples

Example

H_1 is colored 0. So $f(x,x)$ is always 0. (x,y) where $x=y$.

H_2 is colored 1. So $f(x,y)$ with $x<y$ is always 1.

H_3 is colored **R**. So $f(x,y)$ with $x>y$ is never in $\{0,1,2\}$

If f is restricted to $D \times D$ then 2 is not hit. So not onto.

Other cases are similar.

Examples

Example

H_1 is colored 0. So $f(x,x)$ is always 0. (x,y) where $x=y$.

H_2 is colored 1. So $f(x,y)$ with $x<y$ is always 1.

H_3 is colored **R**. So $f(x,y)$ with $x>y$ is never in $\{0,1,2\}$

If f is restricted to $D \times D$ then 2 is not hit. So not onto.

Other cases are similar.

Why 4? We will discuss this more after we do the Thin Set Theorem for $f(x,y,z)$.
(We will do this after the midterm.)