

BILL, RECORD LECTURE!!!!

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Well Quasi Orders

Exposition by **William Gasarch**

March 26, 2026

Motivating A Definition

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Def An ordering (X, \preceq) is **javier** if the following hold

- 1) there are no $\infty \downarrow$ subsequences,
- 2) there are no ∞ antichains.

Well Quasi Orders

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there exists i, j such that $i < j$ and $x_i \preceq x_j$.

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Note We show that (X, \preceq) is a **wqo** iff (X, \preceq) is **javier**.

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So color is UP.

Bonus We don't just get an uptick, we get an $\infty \uparrow$ sequence.

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Let (X, \preceq) be wqo. We show (X, \preceq) is javier.

Let x_1, x_2, x_3, \dots be a sequence of elements of X .

Since the sequence has an uptick it **is not** $\infty \downarrow$ and not an ∞ antichain.

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The following are equivalent

1. For every sequence x_1, x_2, \dots there is an uptick.
2. For every sequence x_1, x_2, \dots there is an increasing subsequence.
3. No sequence x_1, x_2, \dots is an $\infty \downarrow$ or an ∞ anti-chain.

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Lemma If (X, \preceq_1) and (Y, \preceq_2) are wqo then $(X \times Y, \preceq)$ is a wqo.

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Lemma If (X, \preceq_1) and (Y, \preceq_2) are wqo then $(X \times Y, \preceq)$ is a wqo.

Let $(x_1, y_1), (x_2, y_2), (x_3, y_3), \dots$ be an ∞ seq of elts from $A \times B$.

Let $\text{COL}(i, j)$ be one of the nine elts in

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$\{ \text{UP}, \text{INC}, \text{DOWN} \} \times \{ \text{UP}, \text{INC}, \text{DOWN} \}$

Homog set must be UP-UP else not wqo.

wqo Closed Under Finite Power Set

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Then $(2^{\text{fin}X}, \preceq')$ is a wqo.

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If there is a tie then pick one arbitrarily

Proof that $2^{\text{fin}\mathbb{N}}$ is a wqo: Bad Seqs

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(Recall that the X_i are all finite subsets of \mathbb{N} .)

Proof that $2^{\text{fin}\mathbb{N}}$ is a wqo: Bad Seqs

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There may be many bad sequences. We form one: next slide.

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$\{3, 5, 10, 12\}, \{10, 12, 33\}, \{1, 90\}, \dots$ (no pattern implied)

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then B_1 is $\{1, 2, 100\}$ or $\{3, 4, 99\}$. I set $B_1 = \{1, 2, 1000\}$.

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B_1 , the smallest set that is the first set of a sequence in BAD.

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We want to take a second set.

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There are some sequences that do not begin with B_1 .

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What do we do with those sequences?

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Thin out BAD by eliminating all sequences in BAD that don't begin with B_1 .

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Keep this up to get B_1, B_2, \dots . This is called a **Min Bad Seq.**

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The sequence B_1, B_2, \dots , is a bad sequence (easy).

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Example

If $B_1 = \{10, 111, 900\}$ then $b_1 = 10$ and $C_1 = \{111, 900\}$.

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Let $XX = \{C_1, C_2, C_3, \dots\}$.

Shaving The Min Bad Sequence

We have B_1, B_2, \dots

All of the B_i are finite sets.

They are all nonempty since if $B_i = \emptyset$ then $B_i \preceq B_{i+1}$.

For $i \in \mathbb{N}$

Let b_i be the least element in B_i .

Let $C_i = B_i - \{b_i\}$.

Example

If $B_1 = \{10, 111, 900\}$ then $b_1 = 10$ and $C_1 = \{111, 900\}$.

Let $XX = \{C_1, C_2, C_3, \dots\}$. This is a **set** not a **sequence**.

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Upshot Can assume that the first index is the least index in the sequence.

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We are not done. We've just proven that XX is a wqo.

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The Kruskal Tree Theorem

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We denote this $H \preceq_m G$. It is an ordering.

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We present the proof of Nash-Williams.

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T_i is the smallest i th element of a bad sequence that begins T_1, \dots, T_{i-1} .

U_i is the **set** of trees you get if you remove the root r_i of T_i .

XX is union of all trees in all the U_i . ROOTS is the set of roots.

Prove XX is a wqo (this is the hard part).

By prior Theorems, $2^{\text{fin}XX} \times \text{ROOTS}$ is a wqo.

Look at sequence $(U_1, r_1), (U_2, r_2), (U_3, r_3), \dots$

There exists $i \leq j$ such that $(U_i, r_i) \preceq (U_j, r_j)$.

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The Graph Minor Theorem

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GMT Let X be the set of graphs. Then (X, \preceq_m) is a wqo.