BILL, RECORD LECTURE!!!!

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Honest Colorings of The Plane

**by Bill Gasarch and
Ryan's friend's friend**

May 2, 2026

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Convention Throughout this talk
Coloring means **Honest Coloring**

Colorings of $\mathbb{R}^2$:

Looking for a-Lines

Coloring $\mathbb{R}^2$ Looking for 2-lines and 3-lines

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More interesting question:
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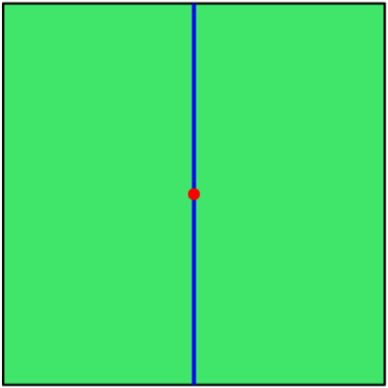
See next slide for 3-coloring of $\mathbb{R}^2$ with no 3-line.

3-Coloring of $\mathbb{R}^2$ with no 3-Line



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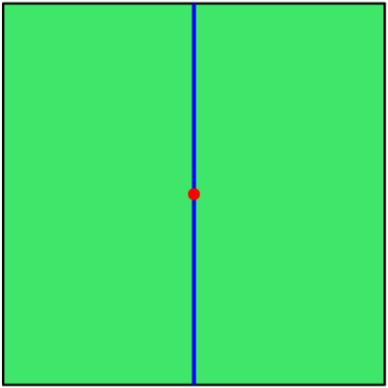
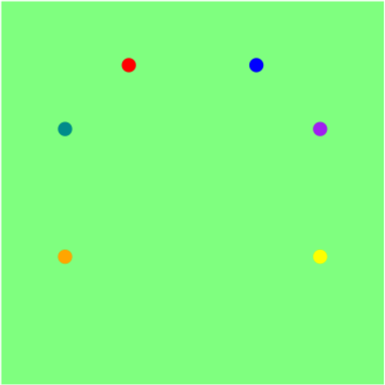


All of the points not on this plane are colored **G** .

7-coloring of $\mathbb{R}^2$ With No 4-Line



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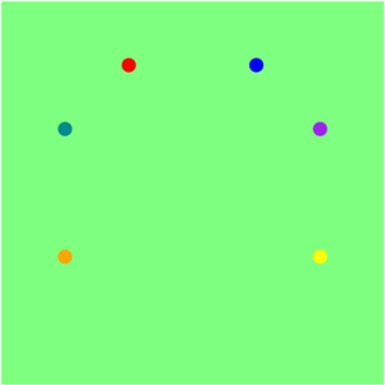

Color as shown:

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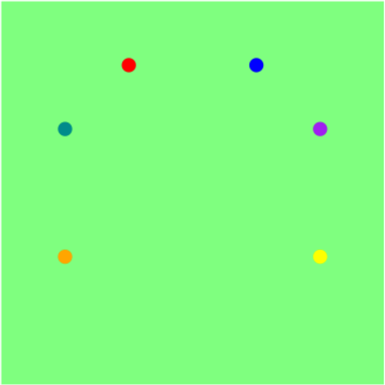

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The 6 points all differently (but not light green).

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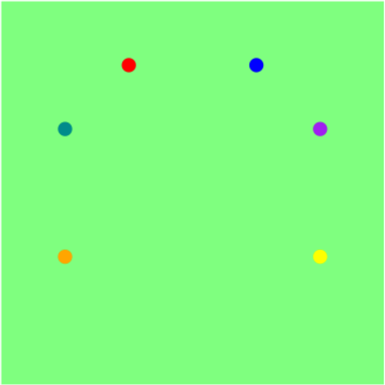

Color as shown:
The 6 points all differently (but not light green).
All the other points of $\mathbb{R}^2$ are light green.

c-coloring of $\mathbb{R}^2$ With No 4-Line

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Color everything else c.

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A 4-line has to use 3 of the points $p_1, \ldots, p_{c-1}$. But no three of those are colinear. So there is no 4-line.

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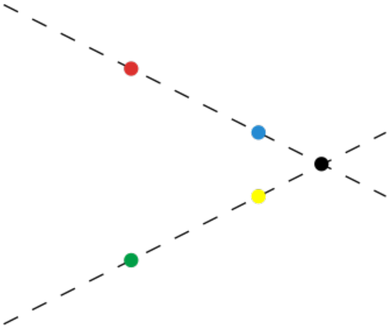
There is a **G** point and a **Y** point. Let L_2 be the line between.

L_1 and L_2 Intersect



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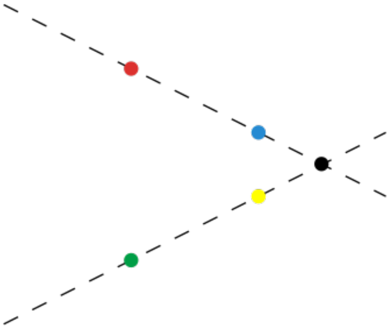
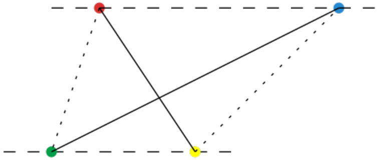


Whatever color the intersection is, there will be a 3-line.

L_1 and L_2 Do Not Intersect



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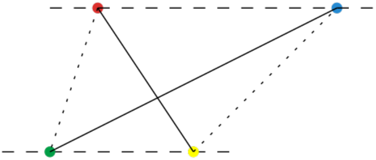

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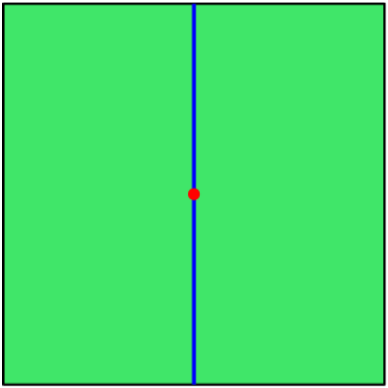

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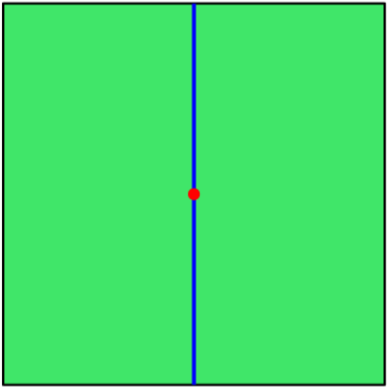

All of the points not on this plane are colored **G** .

4-coloring of $\mathbb{R}^3$

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WORK ON IN GROUPS

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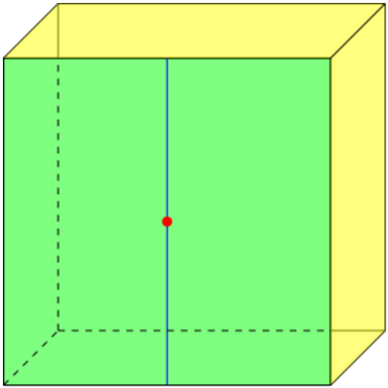

5-coloring of $\mathbb{R}^3$

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Two cases based on if L intersects P or not.

Line L intersects Plane P

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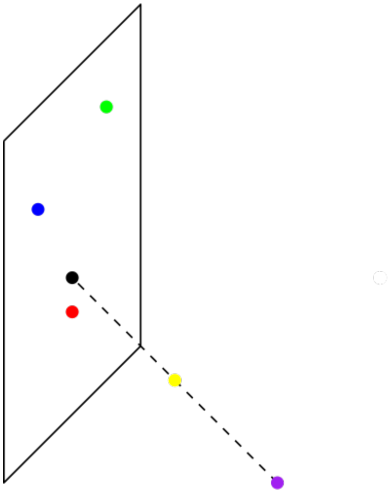

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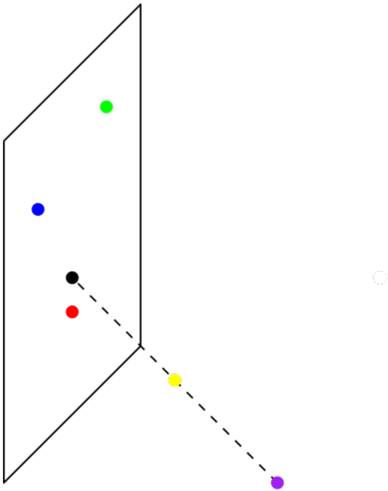

P only has **R**, **B**, and **G**.

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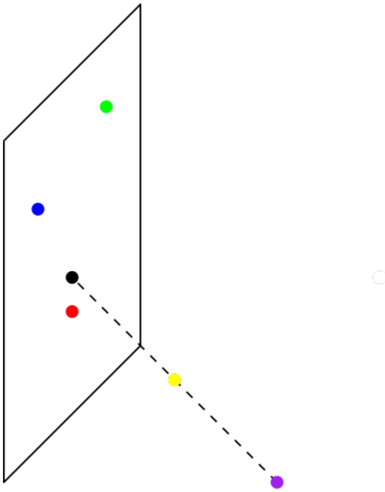

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Point of intersection has to be colored **R**, **B**, or **G**. Say **R**.

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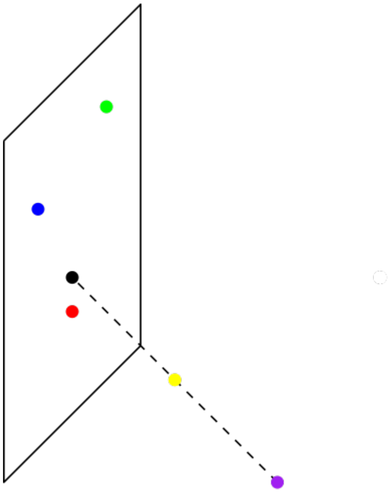

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L has **R**, **Y**, PURPLE. DONE!

Line L Does Not intersects Plane P

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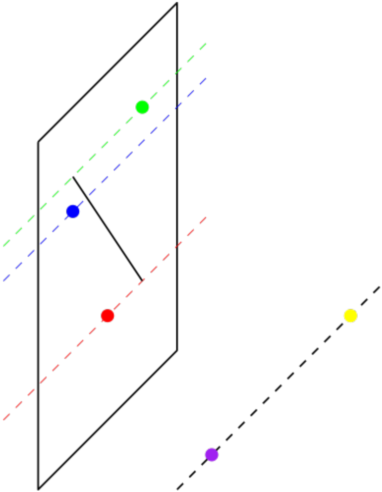

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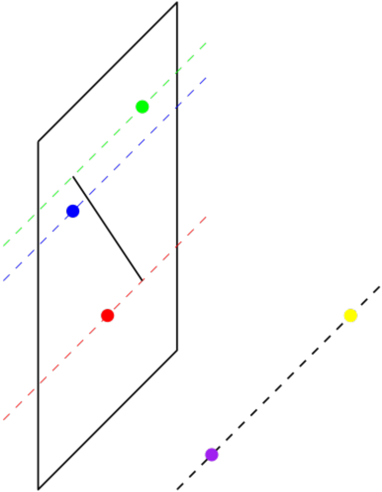

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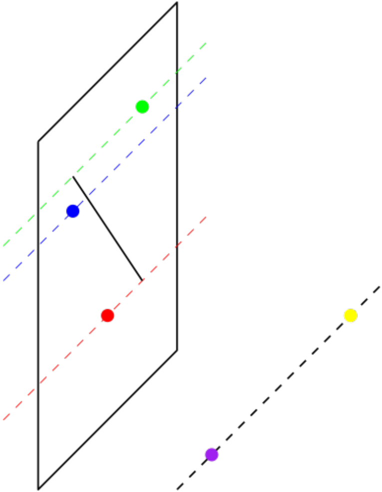
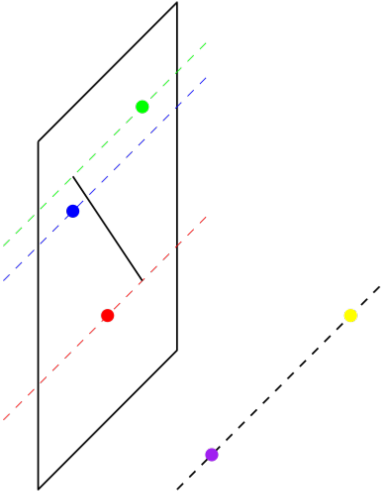


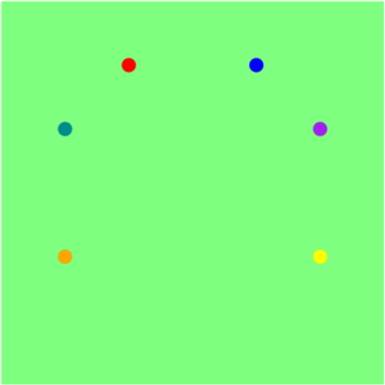
Let Q be an plane through L that intersects P in a line M.
If M has ≥ 2 colors then Q has 4 colors, prior theorem yields 3-line.

Line L Does Not intersects Plane P



Let Q be an plane through L that intersects P in a line M.
If M has ≥ 2 colors then Q has 4 colors, prior theorem yields 3-line.
If all such M are mono then, as in picture, a cross cutting line has 3 colors.

7-coloring of $\mathbb{R}^3$ With No 4-Line



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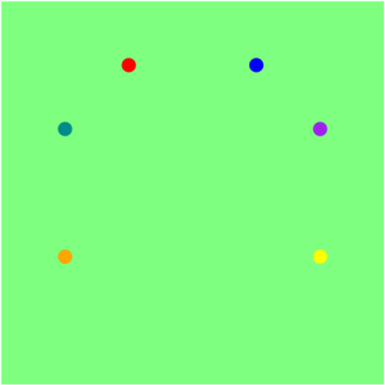

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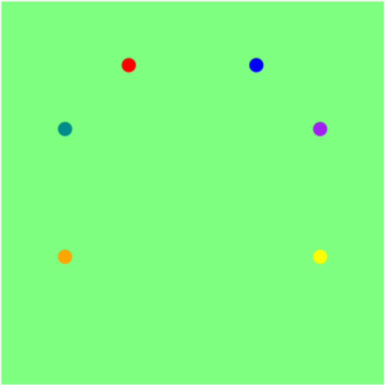

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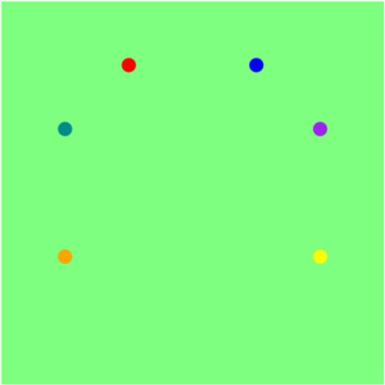

Color as shown, though I have to tell you about the third dim.
The 6 points all differently (but not light green).
All the other points of $\mathbb{R}^3$ are light green.

Summary of Result on Coloring $\mathbb{R}^3$ to get a-Lines

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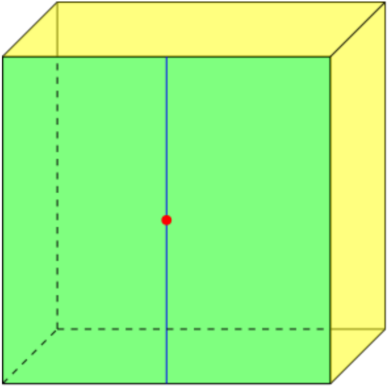
Colorings of $\mathbb{R}^3$:

Looking for a-Planes

$\exists \mathrm{COL}\colon \mathbb{R}^3 \to [4]$ with no 4-plane

$\exists COL\colon \mathbb{R}^3 \rightarrow [4]$ with no 4-plane



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Let the colors be **R**, **B**, **G**, **Y**, **O**.

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If **Y** is on L then you are done.

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If **Y** is not L then let P be the plane that has L on it and the **Y** point on it.

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Let the colors on L be **R**, **B**, **G**.

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Thats a 4-line.

$\forall \mathrm{COL}\colon \mathbb{R}^3 \to [5]\ \exists$ 4-plane

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We have shown that

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Hence
$\forall \mathrm{COL}\colon \mathbb{R}^3 \rightarrow [5]\ \exists$ 4-plane.

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$\forall c \exists \mathrm{COL}\colon \mathbb{R}^3 \to [c]$ with no 5-plane

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Color everything else in $\mathbb{R}^3$ with c.

Summary

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From what we did we could write a summary of colorings of $\mathbb{R}^3$ and a-planes.

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But instead we leave it to you.