

BILL, RECORD LECTURE!!!!

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Well Quasi Orders: Fast Growing Functions

Exposition by **William Gasarch**

March 10, 2026

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We need to prove that $\text{tree}(n)$ always exists.

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We now form an ∞ bad sequence which is a contradiction.

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there is a bad sequence of length L with $|T_i| \leq i + n$.

there is no such bad sequence of length $L + 1$.

What is Known About $tree(n)$?

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tree grows **much faster** than Ackermann's function.

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We can define more levels of a hierarchy by indexing functions by countable ordinals.

Lets look at the countable ordinals

$1, 2, 3, \dots, \omega$

$\omega + 1, \omega + 2, \dots, 2\omega$

$\omega, 2\omega, 3\omega, \dots, \omega^2$.

$\omega, \omega^2, \omega^3, \dots, \omega^\omega$

$\omega\omega^\omega, \omega^{\omega^\omega}, \dots \epsilon_0$

$\text{tree}(n)$ grows roughly like f_{ϵ_0}

TREE: A Faster Growing Functions

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Known:

$\text{TREE}(1) = 1$ $\text{TREE}(2) = 3$ $\text{TREE}(3)$: See Next Slide.

tree vs TREE

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$tree^{tree^{tree^{tree^{tree^8(7)(7)(7)(7)(7)}}}}$

Suffice to say that $TREE(n)$ grows much faster than $tree(n)$.

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Is the function TREE natural? VOTE

Who says YES:

Who says NO:

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gm grows much faster than [tree](#)

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Known: $GM(1)$ is not known.

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where G_i has at most i vertices.

Known: $GM(1)$ is not known.

GM grows much faster than **gm** and **TREE**.

How Come TREE is Better Known Than GM

I asked ChatGPT:

Why is TREE(3) better known than analogous GM(k) functions?

Short answer: GM(k)-type functions are at least as enormous as TREE(k), but they are far less canonical, far less clean, and much harder to package into a single dramatic integer.

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[Bill Here] see next few slides for more ChatGPT commentary and my commentary on their commentary.

Simple vs Complicated Definitions

TREE(3) has a remarkably compact definition:

It is the maximum length of a sequence of finite rooted trees with labels $\{1, 2, 3\}$ such that no tree embeds into a later one.

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[Bill Here] Agree. Note that I can teach KTT in this class but I could not teach GMT in any class.

Proof-Theoretic Position

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[Bill Here] This may be why logicians prefer KTT to GMT but not why the public does.

Shock Value and Simplicity

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Graph minors has the incomprehensible magnitude but lacks that simplicity hook.

You Can Write Down TREE(3) (sort of)

