A Higher Dimensional Version of a Problem of Erdős

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Abstract

Let $\{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d$. We think of $d \ll n$. How big is the largest subset X of points such that all of the distances determined by elements of $\binom{X}{2}$ are different? We show that $|X| \geq \Omega(n^{\frac{1}{3d-3+o(1)}})$. This improves on the best known result which was $|X| \geq \Omega(n^{\frac{1}{3d-2}})$.

Assume that no a of the points are on the same $(a-1)$-hyperplane. How big is the largest subset X of points such that all of the volumes determined by elements of $\binom{X}{a}$ are different? We show that $|X| \geq \Omega(n^{\frac{1}{(2a-1)d}})$. This concept had not been studied before.

Let α be a regular cardinal between $\aleph_0$ and $2^{\aleph_0}$. Let $X \subseteq \mathbb{R}^d$ such that no a of the original points are in the same $(a-1)$-hyperplane. We show that there is an α-sized subset of X such that all of the volumes determined by elements of $\binom{X}{a}$ are different. We give two proofs: one assuming the Axiom of Choice and one assuming the Axiom of Determinacy.

1 Introduction

Let $\{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d$. We think of $d \leq n$. How big is the largest subset X of points such that all of the distances determined by elements of $\binom{X}{2}$ are different? Assume that no three

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of the original points are collinear. How big is the largest subset X of points such that all of the areas determined by elements of $\binom{X}{3}$ are different?

To extend the notion of *no three collinear* we use hyperplanes. We remind the reader of how the dimension works.

Def 1.1 A d-hyperplane is of dimension $d-1$.

Def 1.2 Let $1 \le a \le d+1$. Let $h_{a,d}(n)$ be the largest integer so that if $\{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d$, no a points in the same $(a-1)$-hyperplane, then there exists a subset X of $h_{a,d}(n)$ points for which all of the volumes determined by elements of $\binom{X}{a}$ are different. The $h_{a,d}(n)$*-problem* is to determine (or establish upper and lower bounds on) $h_{a,d}(n)$. The definition extends to letting n be an infinite cardinal α where $\aleph_0 \le \alpha \le 2^{\aleph_0}$.

Below we summarize all that is know about $h_{a,d}(n)$ (to our knowledge).

1. Erdős [5], in 1946, showed that the number of distinct distances in the $\sqrt{n} \times \sqrt{n}$ grid is $\le O(\frac{n}{\sqrt{\log n}})$. Therefore $h_{2,2}(n) \le O\left(\sqrt{\frac{n}{\sqrt{\log n}}}\right)$.

2. Erdős [6], in 1950, showed that, for $\aleph_0 \le \alpha \le 2^{\aleph_0}$, $h_{2,d}(\alpha) = \alpha$. His proof used the Axiom of Choice. He stated that there was a much easier argument if α is regular and this extends to showing $h_{a,d}(\alpha) = \alpha$.

3. Erdős considered the $h_{2,d}(n)$ problem 1957 [7] and 1970 [8]. In the latter paper he notes that $h_{2,2}(7) = 3$ [10] and $h_{2,3}(9) = 3$ [3]. Erdős conjectured that $h_{2,1}(n) = (1+o(n))n^{1/2}$ and notes that $h_{2,1}(n) \le (1+o(n))n^{1/2}$ [11].

4. Komlos, Sulyok and Szemeredi [17], in 1975, show that $h_{2,1}(n) \ge \Omega(\sqrt{n})$ though they state it in different terms.

5. Erdős [9] considered the $h_{2,d}(n)$ problem in 1986. He states *It is easy to see that* $h_{2,d}(n) > n^{\epsilon_d}$ *but the best possible value of* ϵ_d *is not known.* $\epsilon_1 = \frac{1}{2}$ *follows from a result of Ajtai, Komlos, Sulyok and Szemeredi [17].* (We do not know why he added Ajtai since Ajtai was not an author of that paper.)

6. Avis, Erdős, and Pach [1], in 1991, showed that for all sets of n points in the plane, for almost all k-subsets X where $k = o(n^{1/7})$, the elements of $\binom{X}{2}$ determine different distances. Hence, for example, $h_{2,2}(n) = \Omega(n^{1/7+\epsilon})$.

7. Thiele [19], in his PhD thesis from 1995, has as Theorem 4.33, that for all $d \ge 2$, $h_{2,d} = \Omega(n^{\frac{1}{3d-2}})$.

8. Charalambides [2], in 2012, showed that $h_{2,2}(n) = \Omega(n^{1/3}/\log n)$. We use this in the form of $\Omega(n^{\frac{1}{3+o(1)}})$.

9. To our knowledge $h_{a,d}(n)$ for $a \geq 3$ was first studied in an earlier version of this paper [14].

Note 1.3 The problem of $h_{2,2}$ is similar to but distinct from the *Erdős Distance Problem*: given a set of n points in the plane, how many distinct distances are guaranteed? For more on this problem see [12, 13]. The problem of $h_{3,2}$ is similar to but distinct from the problem of determining, given n points in the plane no three collinear, how many distinct triangle-areas are obtained (see [4] and references therein). We do not know of any reference to a higher dimensional analog of these problems.

Notation 1.4 Throughout the paper AC means *The Axiom of Choice*, AD means *The Axiom of Determinacy*, and DC means *The Axiom of Determined Choice*. We assume ZF throughout.

Below we list our results and note which ones are not new. We list them in two categories: Finite and Infinite.

- $h_{2,d}(n) \geq \Omega(n^{\frac{1}{3d-3+o(1)}})$. (Charalambides had the $d = 2$ case.)
- $h_{a,d}(n) \geq \Omega(n^{\frac{1}{(2a-1)d}})$.

We now list all the results known, both ours are others (all by Erdős) on $h_{a,d}(\alpha)$ for infinite α.

- (DC) $h_{a,d}(\aleph_0) = \aleph_0)$. Erdős [6] showed this for $a = 2$ and noted (correctly) that the proof holds for all a.
- (AC) For all $\aleph_0 \leq \alpha \leq 2^{\aleph_0}$, $h_{1,d}(\alpha) = \alpha$. This will fall out of our results; however, it is easy and likely known.
- (AC) For all $\aleph_0 \leq \alpha \leq 2^{\aleph_0}$, $h_{2,d}(\alpha) = \alpha$. Erdős [6] showed this.
- (AC) For all regular α, $\aleph_0 \leq \alpha \leq 2^{\aleph_0}$, $h_{a,d}(\alpha) = \alpha$. Erdős [6] showed this for $a = 2$ and said this proof extended to all a. We do not know his proof; however, we give a proof.
- (AD+DC) Under AD note that their are no cardinalities strictly between $\aleph_0$ and $2^{\aleph_0}$. Also note that $2^{\aleph_0}$ is regular (though we do not use this). We have $h_{a,d}(2^{\aleph_0}) = 2^{\aleph_0})$.

The following definition will permeate the entire paper:

Def 1.5 Let $d \geq 1$. Let $X \subseteq \mathbb{R}^d$. Let $1 \leq a \leq d+1$. An *a-maximal subset of X* is a set M such that, (1) all the a-subsets of M have different volumes, and (2) for all $x \in X - M$ the set $M \cup \{x\}$ does not have this property. We may refer to this as a *maximal subset of X* if a is understood.

2 $h_{2,d}(n) \geq \Omega(n^{\frac{1}{3d-3+o(1)}})$

To prove $h_{2,d}(n) \geq \Omega(n^{\frac{1}{3d-3+o(1)}})$ we will first prove a similar theorem for points on a d-dimensional sphere.

Def 2.1 Let $d \in \mathbb{N}$.

1. If $p, q \in \mathbb{R}^d$ then let $|p - q|$ be the Euclidean distance between p and q.
2. The d-sphere with center $x \in \mathbb{R}^{d+1}$ and radius $r \in \mathsf{R}^+$ is the set
$$\{y \in \mathbb{R}^{d+1} : |x - y| = r\}.$$
(This definition is about d-spheres in R^{d+1}; however, we use the obvious notion of a d-sphere in higher dimensions without comment.)

Def 2.2 Let $h'_{2,d}(n)$ be the largest integer so that if $\{p_1, \ldots, p_n\}$ is on a sphere of dimension d, then there exists a subset X of $h'_{2,d}(n)$ points for which all of the distances determined by elements of $\binom{X}{2}$ are different.

The following Lemma is well known.

Lemma 2.3 *Let S be a d-sphere. Let $y, z \in S$ and $r \in \mathsf{R}^+$.*

1. *The set*
$$\{x \in S : |x - y| = r\}$$
is either a $(d-1)$-sphere or is empty.
2. *The set*
$$\{x \in S : |x - y| = |x - z|\}$$
is a $(d-1)$-sphere.

Theorem 2.4 *For $d \geq 2$, $h'_{2,d}(n) \geq \Omega(n^{\frac{1}{3d-3+o(1)}})$.*

Proof:
We prove this by induction on d.
Base Case: $d = 2$. Charalambides [2] showed that $h'_{2,2}(n) = \Omega(n^{\frac{1}{3+o(1)}})$.

Induction Step: Assume the theorem is true for $d - 1$. We prove it for d.
Let X be a subset of a sphere of dimension d. Let M be a maximal subset of X.
Let $x \in X - M$. Why is $x \notin M$? One of the following must occur:

1. There exists $u \in M$ and $\{u_1, u_2\} \in \binom{M}{2}$ such that $|x - u| = |u_1 - u_2|$.

2. There exists $\{u_1, u_2\} \in \binom{M}{2}$ such that $|x - u_1| = |x - u_2|$.

We map $X - M$ to $M \times \binom{M}{2} \cup \binom{M}{2}$ by mapping $x \in M - X$ to either $(u, \{u_1, u_2\})$ or $\{u_1, u_2\}$ as indicated above.

There are two cases based on a parameter δ which we pick later.

Case 1: All elements in the co-domain have inverse image of size at most n^δ. Then the map is at most n^δ-to-1. Hence

$$|X - M| \le n^\delta \left| M \times \binom{M}{2} \cup \binom{M}{2} \right|.$$

Recall that $|X| = n$. Let $|M| = m$. We can assume $m \ll n$.

$$n \le O(n^\delta m^3)$$

$$m \ge \Omega(n^{\frac{1-\delta}{3}}).$$

Case 2: Some element of the co-domain has inverse image of size at least n^δ. By Lemma 2.3 the inverse image of a an element of the co-domain is a sphere of dimension $d-1$. Hence we apply the theorem inductively to the inverse image to get a set of size $\Omega(n^{\frac{\delta}{3(d-1)-3+o(1)}})$.

Take $\delta \sim \frac{d-1}{d}$ to obtain $\Omega(n^{\frac{1}{3d-3+o(1)}})$ in both cases.

∎

Theorem 2.5 *For $d \ge 2$, $h_{2,d}(n) \ge \Omega(n^{\frac{1}{3d-3+o(1)}})$.*

Proof:

We prove this by induction on d.

Base Case: $d = 2$. Charalambides [2] showed that $h_{2,2}(n) = \Omega(n^{\frac{1}{3+o(1)}})$.

Induction Step: Assume the theorem is true for $d - 1$. We prove it for d.

Let X be a subset of $\mathbb{R}^d$. Let M be a maximal subset of X. Let $m = |M|$.

Let $x \in X - M$. Why is $x \notin M$? One of the following must occur:

1. There exists $u \in M$ and $\{u_1, u_2\} \in \binom{M}{2}$ such that $|x - u| = |u_1 - u_2|$.

2. There exists $\{u_1, u_2\} \in \binom{M}{2}$ such that $|x - u_1| = |x - u_2|$.

We map $X - M$ to $M \times \binom{M}{2} \cup \binom{M}{2}$ by mapping $x \in M - X$ to either $(u, \{u_1, u_2\})$ or $\{u_1, u_2\}$ as indicated above.

There are two cases based on a parameter δ which we pick later.

Case 1: All elements in the co-domain have inverse image of size at most n^{δ}. Then the map is at most n^{δ}-to-1. This case is identical to Case 1 of Theorem 2.4 and hence yields $m \geq \Omega(n^{\frac{1-\delta}{3}})$.
Case 2: Some element of the co-domain has inverse image of size at least n^{δ}. There are two cases.

1. The element of the co-domain with a large inverse image is $\{u_1, u_2\}$. The set of points that map to $\{u_1, u_2\}$ is
$$X' = \{x \mid |x - u_1| = |x - u_2|\}.$$
This set is contained in (essentially) $\mathbb{R}^d$. Apply the induction hypothesis to X' to obtain a desired set of size $\Omega(n^{\frac{\delta}{3(d-1)-3+o(1)}})$.

2. The element of the co-domain with a large inverse image is $(u, \{u_1, u_2\})$. The set of points that map to $(u, \{u_1, u_2\})$ is
$$X' = \{x \mid |x - u_1| = |u_1 - u_2|\}.$$
This set is contained in a $(d-1)$-sphere. Apply Theorem 2.4 to obtain a desired set of size $\Omega(n^{\frac{\delta}{3(d-1)-3+o(1)}})$.

Take $\delta \sim \frac{d-1}{d}$ to obtain $\Omega(n^{\frac{1}{3d-3+o(1)}})$ in both cases. ∎

Note 2.6 The proofs of Theorems 2.4 and 2.5 both used a hard Theorem of Charalambides. Hence neither result is elementary. We can obtain an elementary proof of the weaker theorem $(\forall d \geq 1)[h'_{2,d}(n) \geq \Omega(n^{\frac{1}{3d}})]$ as follows: For the $d = 1$ case do a proof similar to the induction step. You will end up with a mapping which is $\leq$ 2-to-1 and hence can take $\delta = 1$. The rest of the proof is identical. We can obtain an elementary proof of the weaker theorem $(\forall d \geq 1)[h_{2,d}(n) \geq \Omega(n^{\frac{1}{3d}})]$ in a similar manner. Note that we need the lower bound on $h'_{2,d}(n)$ to obtain the lower bound on $h_{2,d}(n)$.

3 $h_{a,d}(n) \geq \Omega(n^{\frac{1}{(2a-1)d}})$

To prove $h_{a,d} \geq \Omega(n^{\frac{1}{(2a-1)d}})$ we need to prove a much more general theorem about points on a variety. Hence we will need concepts and results from algebraic geometry. In particular we will need the notions of dimension and degree of algebraic sets. For the most part these have the intuitive meanings, although there are certain anomalies and we will sometimes refer to these intuitive notions without full proofs of the relevant algebra.

For the purposes of these definitions, all varieties will be taken over $\mathbb{P}^N$ (projective N-space over the complex field $\mathbb{C}$). However, the original set of points will come from $\mathbb{R}^N$, which is embedded in $\mathbb{P}^N$ in the obvious way. Note that we distinguish between the ambient dimension N and the dimension of the variety $d \leq N$. When we mention a variety of dimension d we will assume it is in $\mathbb{P}^N$ for some $d \leq N$. This N will never matter and hence will not be mentioned.

In $\mathbb{R}^d$ the geometric notions such as area and volume can all be defined in terms of determinants. This yields polynomials. When we are working over the projective space, we cannot actually define the volume of a set of points. However, in this section, we will only need to track whether certain sets of points have the same volume as another set of points. For $x_1, \ldots, x_d, y_1, \ldots, y_d \in \mathbb{P}^N$ we *define* $VOL(x_1, \ldots, x_d) = VOL(y_1, \ldots, y_d)$ to be true iff the determinants are equal. This can be expressed as the zero of a *homogenous* polynomial. This is important since the definition is now well defined over the projective space. That is, if you replace $x_1, \ldots, x_d, y_1, \ldots, y_d$ with the equivalent points $\lambda_1 x_1, \ldots, \lambda_d x_d, \gamma_1 y_1, \ldots, \gamma_d y_d$ the answer to the volume-equal question does not change.

We will need the following concepts from Algebraic Geometry.

Def 3.1 A *variety* V is defined as the set of solutions in $\mathbb{P}^N$ of polynomials $f_1(x) = \cdots = f_k(x) = 0$. We say V is *irreducible* if it cannot be written as $V = C_1 \cup C_2$, where C_1, C_2 are distinct, non-empty closed sets in $\mathbb{P}^N$.

To every variety there is an invariant referred as the *Hilbert polynomial*. The *dimension* of V is the degree of the Hilbert polynomial of V, and its *degree* is defined to be $d!$ times the leading coefficient of the Hilbert polynomial of V. We will not define the Hilbert polynomial here; even stating the definition requires deep theorems from algebraic geometry.

If the polynomials $f_1, \ldots, f_k$ are well-behaved, then the dimension is $d = n - k$ and the degree of V is the product of the degrees of $f_1, \ldots, f_k$.

A *hyperplane* is a special case of a variety, when the polynomials $f_1, \ldots, f_k$ are linear.

For our purposes, we only need a few key theorems from algebraic geometry. The main lemma needed follows from Theorem I, 7.7 of Hartshorne's book on Algebraic Geometry [15]:

Lemma 3.2 *Let V be an irreducible variety of dimension d and f be a homogeneous polynomial. Let*

$$W = V \cap \{x \mid f(x) = 0\}.$$

Either $W = V$ or all of the following must hold:

1. *W is the union of irreducible varieties $W = Z_1 \cup \cdots \cup Z_j$.*

2. *The degrees of $Z_1, \ldots, Z_j$ are bounded by a function of the degree of V and the degree of f.*

3. *The number of components j is bounded by a function of the degree of V and the degree of f.*

4. *All these components $Z_1, \ldots, Z_j$ have dimension exactly $d - 1$.*

Note that if $d = 1$, then this gives a form of Bezout's theorem: the intersection W consists of components of dimension 0 and bounded degree, that is, a bounded number of isolated points.

Def 3.3 Let $1 \le a \le d+1$. Let $h_{a,d,r}(n)$ be the largest integer so that if $\{p_1,\ldots,p_n\} \subseteq V$ where V is a variety of dimension d and degree r, no a points in the same $(a-1)$-hyperplane, then there exists a subset X of $h_{a,d,r}(n)$ points for which all of the volumes determined by elements of $\binom{X}{a}$ are different.

Note 3.4 It makes no sense to define $h_{3,1}(n)$ since any set of at least three points on the line will have three collinear. But note that $h_{3,1,r}(n)$ does make sense.

In the following theorem, we will not carefully track the degree. However, it is critical that the degrees of all polynomials that arise are bounded.

Theorem 3.5 *Let* $1 \le a \le d+1$. *Let* $r \in \mathbb{N}$. *There is a constant* $A_{a,d,r}$ *such that* $h_{a,d,r}(n) \ge A_{a,d,r} n^{\frac{1}{(2a-1)n}}$.

Proof:

This is a proof by induction on d. The base case $d = 1$ and the inductive step begin in the same way.

Let $X \subseteq V$ be of size n such that no a points lie on the same $(a-1)$-hyperplane. Let M be an a-maximal subset of X. Let $m = |M|$.

Let $x \in X - M$. Why is $x \notin M$? One of the following must occur:

1. There exists $\{u_1,\ldots,u_{a-1}\} \in \binom{M}{a-1}$ and $\{v_1,\ldots,v_a\} \in \binom{M}{a}$ such that
$$VOL(x,u_1,\ldots,u_{a-1}) = VOL(v_1,\ldots,v_a).$$
2. There exists $\{u_1,\ldots,u_{a-1}\} \in \binom{M}{a-1}$ and $\{v_1,\ldots,v_{a-1}\} \in \binom{M}{a-1}$ such that
$$VOL(x,u_1,\ldots,u_{a-1}) = VOL(x,v_1,\ldots,v_{a-1}).$$

We map $X - M$ to $\binom{M}{a-1} \times \binom{M}{a} \cup \binom{M}{a-1} \times \binom{M}{a-1}$ by mapping $x \in M - X$ to either $(\{u_1,\ldots,u_{a-1}\},\{v_1,\ldots,v_a\})$ or $(\{u_1,\ldots,u_{a-1}\},\{v_1,\ldots,v_{a-1}\})$ as indicated above.

Let I be an element of the co-domain. What does the inverse image of I look like? Using the determinant-definition of volume, all points in the inverse image of I are contained in a set of the form $W_I = V \cap \{x \mid f(x) = 0\}$, where f is the polynomial which is zero when (for example) $VOL(x,u_1,\ldots,u_{a-1}) = VOL(v_1,\ldots,v_a)$. Although this volume only makes sense geometrically in $\mathbb{R}^N$, it is still a polynomial. So the object we are interested in is the zero of a polynomial; this can be lifted to a degree-2 homogeneous polynomial over $\mathbb{P}^N$.

We claim that, for all I, we have $W_I \ne V$. For, suppose $W_I = V$. Hence ALL elements of V satisfy the volume condition. There are two subcases

1. $I = (\{u_1,\ldots,u_{a-1}\},\{v_1,\ldots,v_a\})$. Hence all $x \in V$ satisfy $VOL(x,u_1,\ldots,u_{a-1}) = VOL(v_1,\ldots,v_a)$. We can take $x = u_1$ (even though this was not the intention of the map) to obtain $0 = VOL(u_1,u_1,u_2,\ldots,u_{a-1}) = VOL(v_1,\ldots,v_a)$. Hence $v_1,\ldots,v_a$ are on the same $(a-1)$-hyperplane. This is a contradiction.

2. $I = (\{u_1, u_2\}, \{u_3, u_4\})$. This case is similar.

As $W_I \neq V$ for all I, we can apply Lemma 3.2 to get much information about the structure of W_I: for each I, we have $W_I = Z_{I,1} \cup \cdots \cup Z_{I,j_I}$ where $Z_{I,1}, \ldots, Z_{I,j_I}$ are irreducible varieties of degree $\leq r'$ and dimension $d-1$. Note that j_I and r' are bounded by a function of d, r.

At this point, we distinguish between the base case when $d = 1$ and the inductive step $d > 1$. For the base case, all these components have dimension 0. Hence they are all isolated points. So in this case, every point in the co-domain has an inverse image which is bounded in size. As the co-domain has size roughly m^{2a-1} and the domain has size n, this implies that

$$m \geq \Omega(n^{\frac{1}{2a-1}})$$

as desired.

We now turn to the case when $d \geq 2$. There are two sub-cases:

Case 1: There exists I such that one of the irreducible components of W_I has $\geq n^\delta$ points, where δ is a parameter to be chosen.

By Lemma 3.2 this component is an irreducible variety of dimension $d-1$. We now apply the induction hypothesis to obtain a desired set of size $\geq A_{a,d-1,r'} n^{\delta \frac{1}{(2a-1)(d-1)}}$. As r' is bounded by a function of a, d, r this can be regarded as $\Omega(n^{\frac{\delta}{(2a-1)(d-1)}})$ where the multiplicative constant depends only on a, d, r.

Case 2: For all I in the co-domain all components of W_I have $\leq n^\delta$ points of V. By Lemma 3.2 the number of components is bounded by a constant (which may depend on d, r). Hence the mapping has domain of size roughly n, co-domain of size roughly m^{2a-1}, and is $\leq n^\delta$-to-1. Therefore

$$n \leq O(n^\delta m^{2a-1})$$

so

$$m \geq \Omega(n^{\frac{1-\delta}{2a-1}}).$$

Putting cases 1 and 2 together, we have that there is a desired set of size at least

$$\min\{\Omega(n^{\frac{\delta}{(2a-1)(d-1)}}), \Omega(n^{\frac{1-\delta}{2a-1}})\}.$$

Now set $\delta = 1 - 1/d$. This ensures that we have $\Omega(n^{\frac{1}{(2a-1)d}})$ points in our desired set. ∎

Theorem 3.6 $h_{a,d}(n) \geq \Omega(n^{\frac{1}{(2a-1)d}})$.

Proof:

Note that any set of points in $\mathbb{R}^d$ lives in the variety $\mathbb{P}^d$; as shown in Hartshorne's Proposition I, 7.6, this variety has dimension d and degree 1. Hence we have

$$h_{a,d}(n) \geq h_{a,d,1}(n) \geq A_{a,d,1} n^{\frac{1}{(2a-1)d}} = \Omega(n^{\frac{1}{(2a-1)d}}).$$

∎

4 $h_{a,d}(\alpha)$ for Infinite α

We use the following definition throughout

Def 4.1 A cardinal α is *regular* if, for all $\beta < \alpha$ and for all functions $f : \alpha \rightarrow \beta$, there exists an element of the co-domain whose inverse image is of cardinality α.

4.1 AC implies $h_{a,d}(\alpha) = \alpha$ for α Regular

We use $h'_{2,d}$ as defined in Definition 2.2.

The next theorem is due to Erdős; however, our proof seems to be new.

Theorem 4.2 *(AC)*

1. *If* $\aleph_0 \le \alpha \le 2^{\aleph_0}$ *then* $h'_{2,1}(\alpha) = \alpha$.

2. *If* α *is regular and* $\aleph_0 \le \alpha \le 2^{\aleph_0}$ *then, for all* $d \ge 1$, $h'_{2,d}(\alpha) = \alpha$.

3. *If* $\aleph_0 \le \alpha \le 2^{\aleph_0}$ *then* $h_{2,1}(\alpha) = \alpha$.

4. *If* α *is regular and* $\aleph_0 \le \alpha \le 2^{\aleph_0}$ *then, for all* $d \ge 1$, $h_{2,d}(\alpha) = \alpha$.

Proof:

We prove the first two items. The second two are easily obtained by combining the proof techniques of Theorem 2.5 and that of proving the first two items here.

We prove item 2 by induction on d but note that the base case of $d = 1$ does not need α to be regular.

Base Case: $d = 1$. We will do this without assuming regularity. Let X be a subset of the sphere of dimension 1 (a circle). Let M be a maximal subset of X. Let $m = |M|$.

Let $x \in X - M$. Why is $x \notin M$? One of the following must occur:

1. There exists $u \in M$ and $\{u_1, u_2\} \in \binom{M}{2}$ such that $|x - u| = |u_1 - u_2|$.

2. There exists $\{u_1, u_2\} \in \binom{M}{2}$ such that $|x - u_1| = |x - u_2|$.

We map $X - M$ to $M \times \binom{M}{2} \cup \binom{M}{2}$ by mapping $x \in M - X$ to either $(u, \{u_1, u_2\})$ or $\{u_1, u_2\}$ as indicated above. For all $\{u, \{u_1, u_2\}\} \in M \times \binom{M}{2}$ there are at most two $x \in X - M$ that map to it. For all $\{u_1, u_2\} \in \binom{M}{2}$ there is at most two $x \in X - M$ that maps to it. Hence the map is $\le$ 2-to-1. Therefore

$$|X - M| \le 2 \left| M \times \binom{M}{2} \cup \binom{M}{2} \right|.$$

If $m < \alpha$ then this would cause a contradiction. Hence $m = \alpha$.

Induction Step: Assume the theorem is true for $d-1$. We prove it for d. We will need that α is regular.

Let X be a subset of a sphere of dimension d. Let M be a maximal subset of X. By reasoning similar to that of Theorem 2.4 we obtain a map from $X-M$ to $M\times\binom{M}{2}\cup\binom{M}{2}$ where all of the inverse images of elements of the co-domain are empty or spheres of dimension $d-1$. If $|M|<\alpha$ then, since α is regular, some element of the codomain has to have inverse image of size α. Apply the induction hypothesis to that inverse image. ∎

4.2 AC implies $h_{a,d}(\alpha)=\alpha$ for α Regular

Theorem 4.3 *(AC) If $\aleph_0 \le \alpha \le 2^{\aleph_0}$ and α is regular then for all $1 \le a \le d+1$, $h_{a,d}(\alpha)=\alpha$.*

Proof: Let $X \subseteq \mathbb{R}^d$. Let M be a maximal subset of X.

Let $x \in X-M$. Why is $x \notin M$? One of the following must occur:

1. There exists $\{u_1,\ldots,u_{a-1}\} \in \binom{M}{a-1}$ and $\{v_1,\ldots,v_a\} \in \binom{M}{a}$ such that
$$VOL(x,u_1,\ldots,u_{a-1}) = VOL(v_1,\ldots,v_a).$$

2. There exists $\{u_1,\ldots,u_{a-1}\} \in \binom{M}{a-1}$ and $\{v_1,\ldots,v_{a-1}\} \in \binom{M}{a-1}$ such that
$$VOL(x,u_1,\ldots,u_{a-1}) = VOL(x,v_1,\ldots,v_{a-1}).$$

We map $X-M$ to $\binom{M}{a-1}\times\binom{M}{a}\cup\binom{M}{a-1}\times\binom{M}{a-1}$ by mapping $x \in M-X$ to either $(\{u_1,\ldots,u_{a-1}\},\{v_1,\ldots,v_a\})$ or $(\{u_1,\ldots,u_{a-1}\},\{v_1,\ldots,v_{a-1}\})$ as indicated above.

If $|M|<\alpha$ then, since α is regular, there is an element of the co-domain whose cardinality if α. By using the determinant definition of volume this set is an algebraic variety. We call this set M_0.

Apply the same argument to M_0 to either obtain a maximal set of size α (so we are done) or an algebraic variety $M_1 \subset M_0$ of size α. This process must terminate since, if not, then we have an infinite descending chain of algebraic varieties, which violates Hilbert's Basis Theorem. When the process terminates we obtain a maximal set of size α so we are done. ∎

4.3 AD+DC implies $h_{a,d}(\alpha)=\alpha$

For this section we assume AD+DC. Given AD we have that (1) there are no cardinals strictly between $\aleph_0$ and $2^{\aleph_0}$, and (2) $2^{\aleph_0}$ is regular. Also note that the proofs given above for $h_{a,d}(\alpha)=\alpha$, when restricted to $\alpha=\aleph_0$, only used DC, not full AC. Hence the only case of interest under AD+DC is $h_{a,d}(2^{\aleph_0})$.

We use the well-known fact (for example, see [16]) that under *AD*, every uncountable set of reals has a perfect subset. contains a perfect subset.

We will need some other facts that do not use AD or AC (just DC).

4.4 Some Definitions

Define $W_{a,d} : (\mathbb{R}^d)^{a+1}$ via $W_{a,d}(\mathbf{v}_0, \cdots, \mathbf{v}_a) :=$

$$\det\left(\begin{bmatrix} (\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_1) & \cdots & (\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_k) \\ \vdots & \ddots & \vdots \\ (\tilde{\mathbf{v}}_k, \tilde{\mathbf{v}}_1), & \cdots & (\tilde{\mathbf{v}}_k, \tilde{\mathbf{v}}_k) \end{bmatrix}\right)$$

where $\tilde{\mathbf{v}}_i = \mathbf{v}_i - \mathbf{v}_0$.

Let $V_{a,d}(\mathbf{v}_0 \cdots \mathbf{v}_a)$ be the a-ary volume of the simplex with vertices $\mathbf{v}_0 \cdots \mathbf{v}_a$. Then $V_{a,d} = \frac{|\sqrt{W_{a,d}}|}{a!}$. Thus $V_{a,d}(\overline{\mathbf{v}}) = V_{a,d}(\overline{\mathbf{w}})$ iff $W_{a,d}(\overline{\mathbf{v}}) = \pm W_{a,d}(\overline{\mathbf{w}})$ and $V_{a,d}(\overline{\mathbf{v}}) = 0$ iff $W_{a,d}(\overline{\mathbf{v}}) = 0$.

For each $0 \le i \le a$, let $(\overline{\mathbf{v}}, \overline{\mathbf{w}}) \in (\mathbb{R}^d)^{a+1} \times (\mathbb{R}^d)^i = (\mathbb{R}^d)^{a+i+1}$, and let $j \in \{0, 1\}$. Then define

$$P^{i,j}_{a,d}(\overline{\mathbf{v}}, \overline{\mathbf{w}}) := \{\mathbf{z} \in \mathbb{R}^d : W_{a,d}(\overline{\mathbf{v}}) = (-1)^j W_{a,d}(\overline{\mathbf{w}}, \mathbf{v}_i, \cdots, \mathbf{v}_{a-1}, \mathbf{z})\}.$$

For each $0 < i \le a$, let $(\overline{\mathbf{v}}, \overline{\mathbf{w}}) \in (\mathbb{R}^d)^a \times (\mathbb{R}^d)^i$, and let $j \in \{0, 1\}$. Then define

$$Q^{i,j}_{a,d}(\overline{\mathbf{v}}, \overline{\mathbf{w}}) := \{\mathbf{z} \in \mathbb{R}^d : W_{a,d}(\overline{\mathbf{v}}, \mathbf{z}) = (-1)^j W_{a,d}(\overline{\mathbf{w}}, \mathbf{v}_i, \cdots, \mathbf{v}_{a-1}, \mathbf{z})\}.$$

Define $\mathcal{P}_{a,d} = \{P^{i,j}_{a,d}\} \cup \{Q^{i,j}_{a,d}\}$. (Often we will leave a, d implicit.) Given $P \in \mathcal{P}$, define $||P||$ to be the arity of P.

Lemma 4.4 *For any $P \in \mathcal{P}_{a,d}$, and any distinct$(\mathbf{v}_0 \cdots \mathbf{v}_{||P||-1})$ with no $a+1$ points on the same a-hyperplane, $P(\overline{\mathbf{v}})$ does not contain each $\mathbf{v}_i$.*

Proof: Consider first $M := P^{i,j}_{a,d}(\overline{\mathbf{v}}, \overline{\mathbf{w}})$, with distinct inputs. If $i > 0$, then $\mathbf{w}_0 \notin M$, and if $i = 0$, then $\mathbf{v}_0 \notin M$.

Consider second $M := Q^{i,j}_{a,d}(\overline{\mathbf{v}}, \overline{\mathbf{w}})$. Then $\mathbf{w}_0 \notin M$. ∎

We use the well-known fact (for example, see [16]) that under AD, every uncountable set of reals has a perfect subset.

We need a lemma—a special case of a more general (but not more difficult) theorem proved by Mycielski [18]:

Lemma 4.5 *Let A be any complete metric space without isolated points, and let $R \subset A^d$ be nowhere dense in A^d, for some $d \in \omega$. Then there is a perfect $X \subset A$ such that for all distinct $x_1 \cdots x_d \in X$, $(x_1 \cdots x_d) \notin R$.*

Proof: We inductively build an $f : 2^{<\omega} \to \mathcal{P}(A)$, satisfying the following:

1. For each $\mathbf{s} \in 2^{<\omega}$, $f(\mathbf{s})$ is closed with nonempty interior, and has diameter at most $1/|\mathbf{s}|$.

2. If $\mathbf{s} \subset \mathbf{t}$ then $f(\mathbf{t}) \subset f(\mathbf{s})$, and if $\mathbf{s}$ and $\mathbf{t}$ are incompatible, then $f(\mathbf{s}) \cap f(\mathbf{t}) = \emptyset$.

3. For each $m \geq n$ and distinct $\mathbf{s}_1 \cdots \mathbf{s}_d \in 2^{<\omega}$ all of length m, $f(\mathbf{s}_1) \times \cdots \times f(\mathbf{s}_d) \cap R = \emptyset$.

To satisfy the third requirement, list all such $\mathbf{s}_1 \cdots \mathbf{s}_d$ and deal with each in turn, using the fact that R is nowhere dense.

With f in hand, define $X = \bigcup_{\mathbf{x} \in 2^\omega} \bigcap_{n \in \omega} f(\mathbf{x}|n)$. This works. ∎

Lemma 4.6 *Let $1 \leq a \leq d+1$. For any perfect $X \subset \mathbb{R}^d$ with no $a+1$ points on the same a-hyperplane, there is a perfect $Y \subset X$ such that for any $P \in \mathcal{P}$, and for any distinct $(\mathbf{v}_0 \cdots \mathbf{v}_{||P||-1}) \subset X^{||P||}$, $P(\overline{\mathbf{v}})$ is nowhere dense in X.*

Proof: Suppose not; let X_0 be a counterexample. Pick $P_0(\mathbf{v})$ that is not nowhere dense X_0. Then since $M_0 := P_0(\overline{\mathbf{v}})$ is closed, $X_0 \cap M_0$ contains a perfect subset X_1. X_1 is a proper subset of X_0 since it cannot contain each $\mathbf{v}_i$. Continuing in this fashion, we get an infinite descending chain $M_0 \supset M_1 \supset M_2$ of algebraic varieties. This contradicts Hilbert's Basis Theorem; however, we have to show that $M_i \neq M_{i+1}$. This is similar to the $W_I = V$ case of Theorem 3.5. ∎

Theorem 4.7 *(AD+DC) $h_{a,d}(2^{\aleph_0}) = 2^{\aleph_0}$.*

Proof: Let X be an uncountable subset of $\mathbb{R}^d$ with no $a+1$ points on the same a-hyperplane. Since we are assuming AD we can pass to a subset of X that is perfect and rename that X.

By Lemma 4.6 there is a perfect $Y \subset X$, satisfying that for any $P \in \mathcal{P}$, and for any distinct $(\mathbf{v}_0 \cdots \mathbf{v}_{||P||-1}) \subset X^{||P||}$, $P(\overline{\mathbf{v}})$ is nowhere dense in Y. For each $0 \leq i \leq a$ and each $j \in \{0,1\}$, define $F^{i,j}(\mathbf{v}_0 \cdots, \mathbf{v}_a, \mathbf{w}_i, \cdots, \mathbf{w}_a) = W(\mathbf{v}_0, \cdots, \mathbf{v}_a) + (-1)^j W(\mathbf{v}_0, \cdots, \mathbf{v}_{i-1}, \mathbf{w}_i, \cdots, \mathbf{w}_a)$. Let $R^{i,j}$ be the zeros of $F^{i,j}$—a relation over $\mathbb{R}^d$, say of arity $\mathcal{R}(i,j)$. Then it suffices by the lemma to show that each $R^{i,j}$ is nowhere dense in $Y^{\mathcal{R}(i,j)}$. But this follows from the above. ∎

5 Open Questions

For the finite case the obvious open problem is to is to improve our lower bounds and obtain some upper bounds. One day they may match!

For the infinite case there are two open questions

1. What is $h_{a,d}(\alpha)$ for α singular? As a subquestion, what axioms will be needed to prove results (e.g., AC, AD, DC)?

2. (AD) We have that, from both AC and AD, if $2^{\aleph_0}$ is regular, then $h_{a,d}(2^{\aleph_0}) = 2^{\aleph_0}$. What if we have neither AC or AD? Is there a model of ZF - AC - AD where this is false? Or perhaps $h_{a,d}(2^{\aleph_0}) = 2^{\aleph_0}$) implies that one of AC or AD holds.

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