Computability Theory and Ramsey Theory

An Exposition by William Gasarch

All of the results in this document are due to Jockusch [1].

1 A Computable Coloring with NO Infinite c.e. Homog Sets

All of the results in this

Notation 1.1

1. $M_1, M_2, \ldots$ is a standard list of Turing Machines.

2. Note that from e we can extract the code for M_e.

3. $M_{e,s}(x)$ means that we run M_e for s steps.

4. W_e is the domain of M_e, that is,

$$W_e = \{x \mid (\exists s)[M_{e,s}(x) \downarrow].$$

Note that $W_1, W_2, \ldots$ is a list of ALL c.e. sets.

5.

$$W_{e,s} = \{x \mid M_{e,s}(x) \downarrow\}.$$

Theorem 1.2 *There exists a computable* $COL : \binom{N}{2} \to [2]$ *such that there is NO infinite c.e. homog set.*

Proof: We construct $COL : \binom{N}{2} \to [2]$ to satisfy the following requirements (NOTE- *requirements* is the most important word in computability theory.)

$$R_e : W_e \text{ infinite } \implies W_e \text{ NOT a homog set .}$$

CONSTRUCTION OF COLORING

Stage 0: COL is not defined on anything.

Stage s: We define $COL(0, s), \ldots, COL(s-1, s)$. For $e = 0, 1, \ldots, s$:

If this occurs:

$$(\exists x, y \le s-1)[x, y \in W_{e,s} \wedge COL(x, s), COL(y, s) \text{ undefined }]$$

then take the LEAST two x, y for which this is the case and do the following:

- $COL(x, s) = RED$
- $COL(y, s) = BLUE$.

(Note that IF $s \in W_e$ (which we do not know at this time) then R_e would be satisfied.)

After you to through all of the $0 \le e \le s$ define all other $COL(x, y)$ where $0 \le x < y \le s$ that have not been defined by $COL(x, y) = RED$. This is arbitrary. The important things is that ALL $COL(x, s)$ where $0 \le x \le s-1$ are now defined. This is why COL is computable— at stage s we have defined all $COL(x, y)$ with $0 \le x < y \le s$.

END OF CONSTRUCTION

We show that each requirement is eventually satisfied.

For pedagogue we first look at R_1.

If W_1 is finite then R_1 is satisfied.

Assume W_1 is infinite. We show that R_1 is satisfied. Let $x < y$ be the least two elements in W_1. Let s_0 be the least number such that $x, y \in W_{1,s_0}$ Note that, for ALL $s \ge s_0$ you will have

$COL(x, s) = RED$

$COL(y,s) = BLUE$

Since W_1 is infinite there is SOME $s \ge s_0$ with $s \in W_e$. Hence $x, y, s \in W_1$ and show that W_1 is NOT homogenous.

Can we show R_2 is satisfied the same way? Yes but with a caveat- we won't use the least two elements of W_2. We'll use the least two elements of W_2 that are bigger than the least two elements of W_1. We now do this rigorously and more generally.

Claim: For all e, R_e is satisfied:

Proof: Fix e. If W_e is finite then R_e is satisfied.

Assume W_e is infinite. We show that R_e is satisfied. Let $x_1 < x_2 < \cdots < x_{2e}$ be the first (numerically) $2e$ elements of W_e. Let s_0 be the least number such that

- $x_1, \ldots, x_{2e} \in W_{e,s_0}$, and
- For all $x \in \{x_1, \ldots, x_{2e}\}$, for all $1 \le i \le e-1$, if $x \in W_i$ then $x \in W_{i,s_0}$.

KEY: for all $s \ge s_0$, during stage s, the requirements $R_1, \ldots, R_{e-1}$ may define $COL(x,s)$ for some of the $x \in \{x_1, \ldots, x_{2e}\}$. But they will NOT define $COL(x,s)$ for ALL of those x. Why? Because R_i only defines $COL(x,s)$ for at most TWO of those x's, and there are $e-1$ such i, so at most $2e-2$ of those x's have $COL(x,s)$ defined. Hence there will exist x, y such that R_e gets to define $COL(x,s)$ and $COL(y,s)$. Furthermore, they will always be the SAME x, y since the R_i with $i < e$ have already made up their minds about the x in $\{x_1, \ldots, x_{2e}\}$.

UPSHOT: There exists $x, y \in W_e$ such that, for all $s \ge s_0$,

$COL(x,s) = RED$

$COL(y,s) = BLUE$

Since W_e is infinite there is SOME $s \ge s_0$ with $s \in W_e$. Hence $x, y, s \in W_e$ and show that W_e is NOT homogenous.

∎

2 A Computable Coloring with NO c.e.-in-K Homog Sets

Notation 2.1

1. If A is a c.e. set, say A is the domain of M, then A_s is $\{x \le s \mid M_{e,s}(x) \downarrow\}$. Note that, given s, one can compute A_s.

2. $M_1^{()}, M_2^{()}, \ldots$ is a standard list of oracle Turing Machines.

3. Note that from e we can extract the code for $M_e^{()}$.

4. If A is a c.e. set then $M_{e,s}^{A_s}(x)$ means that we run $M_e^{()}$ for s steps and using A_s for the oracle.

5. If A is c.e. then W_e^A is the domain of M_e^A.

$$W_e^A = \{x \mid (\exists s)[M_{e,s}^{A_s}(x) \downarrow].$$

   Note that $W_1^K, W_2^K, \ldots$ is a list of ALL c.e-in-K sets.

6.

$$W_{e,s}^{A_s} = \{x \mid M_{e,s}^{A_s}(x) \downarrow .$$

Theorem 2.2 *There exists $COL : \binom{N}{2} \to [2]$ such that there is NO infinite c.e-in-K. homog set.*

Proof sketch: This will be a HW. But note that its very similar to the proof of Theorem 1.2— if W_e^K is infinite then eventually $W_{e,s}^{K_s}$ will settle down on its first $2e$ elements. ∎

3 A Computable Coloring with NO Σ_2 Homog Sets

We state equivalences of both c.e. and c.e.-in-K. We leave the proofs to the reader.

Theorem 3.1 *Let A be a set. The following are equivalent:*

1. *There exists* e *such that* $A = W_e$. *(*A *is c.e.)*

2. *There exists a decidable* R *such that*

$$A = \{x \mid (\exists y)[(x, y) \in R].$$

(A *is* Σ_1*.)*

3. *There exists* e *such that*

$$A = \{x \mid (\exists y, s)[M_{e,s}(y) = x\}.$$

(This is the origin of the phrase 'computably ENUMERABLE.)

Theorem 3.2 *Let* A *be a set. The following are equivalent:*

1. *There exists* e *such that* $A = W_e^K$. *(*A *is c.e.-in-*K*.)*

2. *There exists a decidable-in-*K R *such that*

$$A = \{x \mid (\exists y)[(x, y) \in R].$$

(A *is* Σ_1^K*.)*

3. *There exists* e *such that*

$$A = \{x \mid (\exists y, s)[M_{e,s}^K(y) = x\}.$$

(This is the origin of the phrase 'computably ENUMERABLE-in-K*.)*

We also need to know that K is quite powerful:

Def 3.3 If A, B are sets then $A \le_m B$ means that there exists a computable f such that

$$x \in A \iff f(x) \in B.$$

We leave the proof of the following to the reader.

Theorem 3.4 *If A is c.e. then $A \leq_m K$.*

The key use of the above theorem is that we can phrase Σ_1 questions as queries to K.

Theorem 3.5 *$A \in \Sigma_2$ iff A is c.e.-in-K.*

Proof:

1) $A \in \Sigma_2$ implies A is c.e.-in-K:

If $A \in \Sigma_2$ then there exists a TM R that always converges such that

$$A = \{x \mid (\exists y)(\forall z)[R(x,y,z) = 1]\}.$$

Let M^K be the TM that does the following:

1. Input(x, y).
2. Ask K $(\forall z)[R(x,y,z) = 1]$. (Can rephrase as $(\exists z)[R(x,y,z) = 0]$.)
3. If YES answer YES, if NO then answer NO.

$$A = \{x \mid (\exists y)[M^K(x,y) = 1]\}.$$

Hence A is c.e.-in-K.

2) A c.e.-in-K implies $A \in \Sigma_2$.

A is c.e.-in-K. So

$$A = W_e^K = \{x \mid (\exists s)(\forall t)[t \geq s \implies x \in W_{e,t}^{K_t}]\}.$$

So A is Σ_2.

∎

Theorem 3.6 *There exists $COL : \binom{N}{2} \rightarrow [2]$ such that there is NO infinite Σ_2 homog set.*

Proof: Combine Theorems 2.2 and 3.5. Note that we only need one part of the implication in Theorem 3.5. ∎

4 Every Computable Coloring has an Infinite Π_2 Homog set

We obtain this with a modification of the usual proof of Ramsey's theorem. the key is that we don't really toss things out- we guess on what the colors are and change our mind.

Theorem 4.1 *For every computable coloring $COL : \binom{N}{2} \rightarrow [2]$ there is an infinite Π_2 homog set.*

Proof:

We are given computable $COL : \binom{N}{2} \rightarrow [2]$.

CONSTRUCTION of $x_1, x_2, \ldots$ and $c_1, c_2, \ldots$.

NOTE: at the end of stage s we might have $x_1, \ldots, x_i$ defined where $i < s$. We will not try to keep track of how big i is. Also, we may have at stage (say) 1000 a sequence of length 50, and then at stage 1001 have a sequence of length only 25. The sequence will grow eventually but do so in fits and starts.

$$x_1 = 1$$

$$c_1 = \text{RED} \quad \text{We are guessing. We might change our mind later}$$

Let $s \geq 2$, and assume that $x_1, \ldots, x_{s-1}$ and $c_1, \ldots, c_{s-1}$ are defined.

1. Ask K *Does there exists $x \geq x_{s-1}$ such that, for all $1 \leq i \leq s-1$, $COL(x_i, x) = c_i$)?*

2. If YES then (using that COL is computable) find the least such x.

$$x_i = \quad x$$

$$c_i = \quad \text{RED We are guessing. We might change our mind later}$$

We have implicitly tossed out all of the numbers between x_{i-1} and x_i.

3. If NO then we ask K how far back we can go. More rigorously we ask the following sequence of questions until we get a YES.

   - *Does there exists* $x \geq x_{s-1}$ *such that, for all* $1 \leq i \leq s-2$, $COL(x_i, x) = c_i$)*?*
   - *Does there exists* $x \geq x_{s-1}$ *such that, for all* $1 \leq i \leq s-3$, $COL(x_i, x) = c_i$)*?*
   - $\vdots$
   - *Does there exists* $x \geq x_{s-1}$ *such that, for all* $1 \leq i \leq 2$, $COL(x_i, x) = c_i$)*?*
   - *Does there exists* $x \geq x_{s-1}$ *such that, for all* $1 \leq i \leq 1$, $COL(x_i, x) = c_i$)*?*

   (One of these must be a YES since (1) if $c_1 = RED$ and there are NO red edges coming out of x_1 then there must be an infinite number of $BLUE$ edges, and (2) if c_1=BLUE its because there are only a finite number of RED edges coming out of x_1 so there are an infinite number of $BLUE$ edges. Let i_0 be such that *There exists* $x \geq x_{s-1}$ *such that, for all* $1 \leq i \leq i_0$, $COL(x_i, x) = c_i$) Do the following:

   (a) Change the color of c_{i+1}. (We will later see that this change must have been from RED to $BLUE$.

   (b) Wipe out $x_{i+2}, \ldots, x_{s-1}$.

   (c) Search for the $x \geq x_{s-1}$ that the question asked says exist.

   (d) x_{i+2} is now x.

(e) c_{i+2} is now RED.

END OF CONSTRUCTION of $x_1, x_2 \ldots$ and $c_1, c_2, \ldots$.

We need to show that there is a Π_2 homog set.

Let X be the set of x_i that are put on the board and stay on the board.

Let R be the set of $x_i \in X$ whose final color is RED.

Claim 1: Once a number turns from RED to $BLUE$ it can't go back to RED again.

Proof:

If a number is turned $BLUE$ its because there are only a finite number of RED edges coming out of it. Hence there must be an infinite number of $BLUE$ edges coming out of it. Hence it will never change color (though it may be tossed out).

End of Proof

Claim 1: $X, R \in \Pi_2$.

Proof:

We show that $\overline{X} \in \Sigma_2$. In order to NOT be in X you must have, at some point in the construction, been tossed out.

$$\overline{X} = \{x \mid (\exists x)[\text{ at stage } s \text{ of the construction } x \text{ was tossed out }]\}.$$

Note that the condition is computable-in-K. Hence $\overline{X}$ is c.e.-in-K. By Theorem 3.5 $\overline{X} \in \Sigma_2$.

We show that $\overline{R} \in \Sigma_2$. In order to NOT be in R you must have to either NOT be in X or have been turned blue. Note that once you turn at some point in the construction, been tossed out.

$$\overline{R} = \overline{X} \cup \{x \mid (\exists x)[\text{ at stage } s \text{ of the construction } x \text{ was turned BLUE}]\}.$$

Recall that Σ_2 is closed under complementation. So we only need to show that the other unionand is in Σ_2. Note that the condition is computable-in-K. Hence $\overline{R}$ is c.e.-in-K. By Theorem 3.5

$\overline{R} \in \Sigma_2$.

End of Proof

There are two cases:

1. If R is infinite then R is an infinite homog set that is Π_2.

2. If R is finite then B is X minus a finite number of elements. Since X is Π_2, B is Π_2.

∎

References

[1] C. Jockusch. Ramsey's theorem and recursion theory. *Journal of Symbolic Logic*, 37(2):268–280, 1972. `http://www/jstor.org/pss/2272972`.