Rectangle Free Coloring of Grids

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Abstract

Let $G_{n,m}$ be the grid $[n]\times[m]$. $G_{n,m}$ is *c-colorable* if there is a function $\chi : G_{n,m} \rightarrow [c]$ such that there are no rectangles with all four corners the same color. We ask *for which values of n, m, c is $G_{n,m}$ c-colorable?* We determine (1) *exactly* which grids are 2-colorable, (2) *exactly* which grids are 3-colorable, (2) *exactly* which grids are 4-colorable. Our main tools are combinatorics and finite fields.

Our problem has two motivations: (1) (ours) A Corollary of the Gallai-Witt theorem states that, for all c, there exists $W = W(c)$ such that any c-coloring of $[W] \times [W]$ has a monochromatic square. The bounds on $W(c)$ are enormous. Our relaxation of the problem to rectangles yields much smaller bounds. (2) Colorings grids to avoid a rectangle is equivalent to coloring the edges of a bipartite graph to avoid a monochromatic $K_{2,2,}$. Hence our work is related to bipartite Ramsey Numbers.

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Contents

1 Introduction

Notation 1.1 If $n \in \mathbb{N}$ then $[n] = \{1, \ldots, n\}$. If $n, m \in \mathbb{N}$ then $G_{n,m}$ is the grid $[n] \times [m]$.

The Gallai-Witt theorem[1] (also called the multi-dimensional Van Der Waerden theorem) has the following corollary: *For all c, there exists $W = W(c)$ such that, for all c-colorings of $[W] \times [W]$ there exists a monochromatic square.* The classical proof of the theorem gives very large upper bounds on $W(c)$. Despite some improvements [2], the known bounds on

[1] It was attributed to Gallai in [15] and [16]; Witt proved the theorem in [23].

$W(c)$ are still quite large. If we relax the problem to seeking a *monochromatic rectangle* then we can obtain far smaller bounds.

Def 1.2 A *rectangle* of $G_{n,m}$ is a subset of the form $\{(a,b),(a+c_1,b),(a+c_1,b+c_2),(a,b+c_2)\}$ for some $a,b,c_1,c_2 \in \mathbb{N}$. A grid $G_{n,m}$ is *c-colorable* if there is a function $\chi : G_{n,m} \to [c]$ such that there are no rectangles with all four corners the same color.

Not all grids have c-colorings. As an example, for any c clearly $G_{c+1,c^{c+1}+1}$ does not have a c-coloring by two applications of the pigeonhole principle. In this paper, we ask the following. Fix c.

For which values of n and m is $G_{n,m}$ c-colorable?

Def 1.3 Let $n,m,n',m' \in \mathbb{N}$. $G_{m,n}$ *contains* $G_{n',m'}$ if $n' \le n$ and $m' \le m$. $G_{m,n}$ *is contained in* $G_{n',m'}$ if $n \le n'$ and $m \le m'$. Proper containment means that at least one of the inequalities is strict.

Clearly, if $G_{n,m}$ is c-colorable, then all grids that it contains are c-colorable. Likewise, if $G_{n,m}$ is not c-colorable then all grids that contain it are not c-colorable.

Def 1.4 Fix $c \in \mathbb{N}$. OBS_c is the set of all grids $G_{n,m}$ such that $G_{n,m}$ is not c-colorable but all grids properly contained in $G_{m,n}$ are c-colorable. OBS_c stands for *Obstruction Sets*.

We leave the proof of the following theorem to the reader.

Theorem 1.5 *Fix $c \in \mathbb{N}$. A grid $G_{n,m}$ is c-colorable iff it does not contain any element of* OBS_c.

By Theorem 1.5 we can rephrase the question of finding which grids are c-colorable:

What is OBS_c*?*

Note that if $G_{n,m} \in \mathrm{OBS}_c$, then $G_{m,n} \in \mathrm{OBS}_c$.

Our problem has another motivation involving the Bipartite Ramsey Theorem which we now state.

Theorem 1.6 *For all L, for all c, there exists n such that for any c-coloring of the edges of $K_{n,n}$ there exists a monochromatic $K_{L,L}$.*

We now state a corollary of the Bipartite Ramsey theorem and a statement about grid colorings that is easily seen to be equivalent to it.

1. For all c there exists n such that for any c-coloring of the edges of $K_{n,n}$ there exists a monochromatic $K_{2,2}$.

2. For all c there exists n such that for any c-coloring of $G_{n,n}$ there exists a monochromatic rectangle.

One can ask, given c, what is n? Beineke and Schwenk [4] studied a closely related problem: What is the minimum value of n such that any 2-coloring of $K_{n,n}$ results in a monochromatic $K_{a,b}$? In their work, this minimal value is denoted $R(a,b)$. Later, Hattingh and Henning [10] defined $n(a,b)$ as the minimum n for which any 2-coloring of $K_{n,n}$ contains a monochromatic $K_{a,a}$ or a monochromatic $K_{b,b}$.

Our results are about $G_{n,m}$ not just $G_{n,n}$ hence they are not quite the same as the bipartite Ramsey numbers. Even so, we do obtain some new Bipartite Ramsey Numbers. They are in Section 8.

The remainder of this paper is organized as follows. In Section 2 we develop tools to show grids *are not* c-colorable. In Section 3 we develop tools to show grids *are* c-colorable. In Section 4 we obtain upper and lower bounds on $|\mathrm{OBS}_c|$. In Section 5, 6, and 7 we find OBS_2, OBS_3, and OBS_4 respectively. In Section 8 we apply the results to find some new bipartite Ramsey numbers. We conclude with some open questions. The appendix contains some sizes of maximum rectangle free sets (to be defined later).

In a related paper, Cooper, Fenner, and Purewal [5] generalize our problem to multiple dimensions and obtain upper and lower bounds on the sizes of the obstruction sets. In another related paper Molina, Oza, and Puttagunta [8] have looked at some variants of our questions.

2 Tools to Show Grids are Not c-colorable

2.1 Using Rectangle Free Sets

A *rectangle-free subset* $A \subseteq G_{n,m}$ is a subset that does not contain a rectangle. A problem that is closely related to grid-colorability is that of finding a rectangle-free subset of maximum cardinality. This relationship is illustrated by the following lemma.

Theorem 2.1 *If $G_{n,m}$ is c-colorable, then it contains a rectangle-free subset of size $\lceil \frac{nm}{c} \rceil$.*

Proof: A c-coloring partitions the elements of $G_{n,m}$ into c rectangle-free subsets. By the pigeon-hole principle, one of these sets must be of size at least $\lceil \frac{nm}{c} \rceil$. ∎

Def 2.2 Let $n, m \in \mathbb{N}$. $\mathrm{maxrf}(n,m)$ is the size of the maximum rectangle-free $A \subseteq G_{n,m}$.

Finding the maximum cardinality of a rectangle-free subset is equivalent to a special case of a well-known problem of Zarankiewicz [24] (see [9] or [18] for more information). The Zarankiewicz function, denoted $Z_{r,s}(n,m)$, counts the minimum number of edges in a bipartite graph with vertex sets of size n and m that guarantees a subgraph isomorphic to $K_{r,s}$. Zarankiewicz's problem was to determine $Z_{r,s}(n,m)$.

If $r = s$, the function is denoted $Z_r(n, m)$. If one views a grid as an incidence matrix for a bipartite graph with vertex sets of cardinality n and m, then a rectangle is equivalent to a subgraph isomorphic to $K_{2,2}$. Therefore the maximum cardinality of a rectangle-free set in $G_{n,m}$ is $Z_2(n, m) - 1$. We will use this lemma in its contrapositive form, i.e., we will often show that $G_{n,m}$ is not c-colorable by showing that $Z_2(n, m) \leq \lceil \frac{nm}{c} \rceil$.

Reiman [17] proved the following lemma. Roman [18] later generalized it.

Lemma 2.3 *Let $m \leq n \leq \binom{m}{2}$. Then $Z_2(n, m) \leq \left\lfloor \frac{n}{2}\left(1 + \sqrt{1 + 4m(m-1)/n}\right)\right\rfloor + 1$.*

Corollary 2.4 *Let $m \leq n \leq \binom{m}{2}$. Let $z_{n,m} = \left\lfloor \frac{n}{2}\left(1 + \sqrt{1 + 4m(m-1)/n}\right)\right\rfloor + 1$ be the upper-bound on $Z_2(n, m)$ in Lemma 2.3. If $z_{n,m} \leq \lceil \frac{nm}{c} \rceil$ then $G_{n,m}$ is not c-colorable.*

Corollary 2.4, and some 2-colorings of grids, are sufficient to find OBS_2. To find OBS_3 and OBS_4, we need more powerful tools to show grids are not colorable (along with some 3-colorings and 4-colorings of grids).

Def 2.5 Let $n, m, x_1, \ldots, x_m \in \mathbb{N}$. $(x_1, \ldots, x_m)$ is (n, m)*-placeable* if there exists a rectangle-free $A \subseteq G_{n,m}$ such that, for $1 \leq j \leq m$, there are x_j elements of A in the j^{th} column.

Lemma 2.6 *Let $n, m, x_1, \ldots, x_m \in \mathbb{N}$ be such that $(x_1, \ldots, x_m)$ is (n, m)-placeable. Then $\sum_{i=1}^{m} \binom{x_i}{2} \leq \binom{n}{2}$.*

Proof: Let $A \subseteq G_{n,m}$ be a set that shows that $(x_1, \ldots, x_m)$ is (n, m)-placeable. Let $\binom{A}{2}$ be the set of pairs of elements of A. Let $2^{\binom{A}{2}}$ be the powerset of $\binom{A}{2}$.

Define the function $f : [m] \to 2^{\binom{A}{2}}$ as follows. For $1 \leq j \leq m$,

$$f(j) = \{\{a, b\} : (a, j), (b, j) \in A\}.$$

If $\sum_{j=1}^{m} |f(j)| > \binom{n}{2}$ then there exists $j_1 \neq j_2$ such that $f(j_1) \cap f(j_2) \neq \emptyset$. Let $\{a, b\} \in f(j_1) \cap f(j_2)$. Then

$$\{(a, j_1), (a, j_2), (b, j_1), (b, j_2)\} \subseteq A.$$

Hence A contains a rectangle. Since this cannot happen, $\sum_{j=1}^{m} |f(j)| \leq \binom{n}{2}$. Note that $|f(j)| = \binom{x_j}{2}$. Hence $\sum_{i=1}^{m} \binom{x_i}{2} \leq \binom{n}{2}$. ∎

Theorem 2.7 *Let $a, n, m \in \mathbb{N}$. Let $q, r \in \mathbb{N}$ be such that $a = qn + r$ with $0 \leq r \leq n$. Assume that there exists $A \subseteq G_{n,m}$ such that $|A| = a$ and A is rectangle-free.*

1. *If $q \geq 2$ then*

$$n \leq \left\lfloor \frac{m(m-1) - 2rq}{q(q-1)} \right\rfloor.$$

2. *If* $q = 1$ *then*
$$r \leq \frac{m(m-1)}{2}.$$

Proof: The proof for the $q \geq 2$ and the $q = 1$ case begins the same; hence we will not split into cases yet.

Assume that, for $1 \leq j \leq m$, the number of elements of A in the j^{th} column is x_j. Note that $\sum_{j=1}^{m} x_j = a$. By Lemma 2.6 $\sum_{j=1}^{m} \binom{x_j}{2} \leq \binom{n}{2}$. We look at the least value that $\sum_{j=1}^{n} \binom{x_j}{2}$ can have.

Consider the following question:
Minimize $\sum_{j=1}^{n} \binom{x_j}{2}$

Constraints:

- $\sum_{j=1}^{n} x_j = a$.
- $x_1, \ldots, x_n$ are natural numbers.

One can easily show that this is minimized when, for all $1 \leq j \leq n$,
$$x_j \in \{\lfloor a/n \rfloor, \lceil a/n \rceil\} = \{q, q+1\}.$$

In order for $\sum_{j=1}^{n} x_j = a$ we need to have $n - r$ many q's and r many $q + 1$'s. Hence we obtain

$\sum_{j=1}^{n} \binom{x_j}{2}$ is at least
$$(n-r)\binom{q}{2} + r\binom{q+1}{2}.$$

Hence we have
$$(n-r)\binom{q}{2} + r\binom{q+1}{2} \leq \sum_{j=1}^{n} \binom{x_j}{2} \leq \binom{m}{2}$$
$$nq(q-1) - rq(q-1) + r(q+1)q \leq m(m-1)$$
$$nq(q-1) - rq^2 + rq + rq^2 + rq \leq m(m-1)$$
$$nq(q-1) + 2rq \leq m(m-1)$$

Case 1: $q \geq 2$.

Subtract $2rq$ from both sides to obtain
$$nq(q-1) \leq m(m-1) - 2rq.$$

Since $q - 1 \neq 0$ we can divide by $q(q-1)$ to obtain
$$n \leq \left\lfloor \frac{m(m-1) - 2rq}{q(q-1)} \right\rfloor.$$

Case 2: $q = 1$.

Since $q - 1 = 0$ we get

$$2r \le m(m-1)$$

$$r \le \frac{m(m-1)}{2}.$$

∎

Corollary 2.8 *Let $m, n \in \mathbb{N}$. If there exists an r where $\frac{m(m-1)}{2} < r \le n$ and $\lceil \frac{mn}{c} \rceil = n + r$, then $G_{m,n}$ is not c-colorable.*

Corollary 2.9 *Let $n, m \in \mathbb{N}$. Let $\lceil \frac{nm}{c} \rceil = qn + r$ for some $0 \le r \le n$ and $q \ge 2$. If $\frac{m(m-1)-2qr}{q(q-1)} < n$ then $G_{n,m}$ is not c-colorable.*

We now show that, for all c, G_{c^2,c^2+c+1} is not c-colorable. This is particularly interesting because by Theorem 3.15, for c a prime power, G_{c^2,c^2+c} is c-colorable.

Corollary 2.10 *For all $c \ge 2$ G_{c^2,c^2+c+1} is not c-colorable.*

Proof: If G_{c^2,c^2+c+1} is c-colorable then there exists a rectangle free subset of G_{c^2,c^2+c+1} of size $\frac{c^2(c^2+c+1)}{c} = c(c^2 + c + 1)$. Let $a = c(c^2 + c + 1)$, $n = c^2 + c + 1$, and $m = c^2$ in Theorem 2.7. Then $q = c$ and $r = 0$. By that lemma we have

$$n \le \left\lfloor \frac{m(m-1) - 2rq}{q(q-1)} \right\rfloor$$

we should have

$$c^2 + c + 1 \le \left\lfloor \frac{c^2(c^2-1)}{c(c-1)} \right\rfloor = c(c+1) = c^2 + c$$

This is a contradiction. ∎

Note 2.11 In the Appendix we use the results of this section to find the sizes of maximum rectangle free sets.

Corollary 2.12

1. *Let $c \ge 2$ and $1 \le c' < c$. Let $n > \frac{c}{c'}\binom{c+c'}{2}$. Then $G_{n,c+c'}$ is not c-colorable.*

2. *Let $c \ge 2$ and $1 \le c' < c$. Let $m > \frac{c}{c'}\binom{c+c'}{2}$. Then $G_{c+c',m}$ is not c-colorable. (This follows immediately from part a.)*

Proof: Assume, by way of contradiction, that $G_{n,c+c'}$ is c-colorable. Then there is a rectangle free set of size

$$\left\lceil \frac{n(c+c')}{c} \right\rceil = \left\lceil n + \frac{c'n}{c} \right\rceil = n + \left\lceil \frac{c'}{c} n \right\rceil .$$

Since $c' < c$ we have

$$\left\lceil \frac{n(c+c')}{c} \right\rceil = n + \left\lceil \frac{c'}{c} n \right\rceil \leq n + \left\lceil \frac{c-1}{c} n \right\rceil = n + \left\lceil n - \frac{n}{c} \right\rceil .$$

The premise of this corollary implies $c < n$. Hence

$$\left\lceil \frac{n(c+c')}{c} \right\rceil \leq n + \left\lceil n - \frac{n}{c} \right\rceil \leq 2n - 1.$$

Therefore when we divide n into $r = \lceil \frac{c'n}{c} \rceil$.

$$\left\lceil \frac{n(c+c')}{c} \right\rceil = n + \left\lceil \frac{c'n}{c} \right\rceil .$$

We want to apply Corollary 2.8 with $m = c + c'$ and $r = \lceil \frac{c'n}{c} \rceil$. We need

$$\frac{m(m-1)}{2} < r \leq n.$$

$$\frac{(c+c')(c+c'-1)}{2} < \left\lceil \frac{c'n}{c} \right\rceil \leq n.$$

The second inequality is obvious. The first inequality follows from $n > \frac{c}{c'}\binom{c+c'}{2}$.

∎

Note 2.13 In the Appendix we use the results of this section to find the sizes of maximum rectangle free sets.

2.2 Using maxrf

Notation 2.14 If $n, m \in \mathbb{N}$ and $A \subseteq G_{n,m}$.

1. We will denote that $(a, b) \in A$ by putting an R in the (a, b) position.

2. For $1 \leq j \leq m$, x_j is the number of elements of A in column j.

3. For $1 \leq j \leq m$, C_j is the set of rows r such that A has an element in the r^{th} row of column j. Formally
$$C_j = \{r : (r, j) \in A\}.$$

Def 2.15 Let $n, m \in \mathbb{N}$ and $A \subseteq G_{n,m}$. Let $1 \le i_1 < i_2 \le n$. C_{i_1} and C_{i_2} *intersect* if $C_{i_1} \cap C_{i_2} \neq \emptyset$.

Lemma 2.16 *Let $n, m \in \mathbb{N}$. Let $x_1 \le n$. Assume $(x_1, \ldots, x_m)$ is (n,m)-placeable via A. Then*

$$|A| \le x_1 + m - 1 + \mathrm{maxrf}(n - x_1, m - 1).$$

Proof: The picture in Table 1 portrays what might happen. We use double lines to partition the grid in a way that will be helpful later.

	1	2	3	4	5	$\ldots$	j	$\cdots$	m
1	R	R				$\cdots$		$\cdots$	
2	R		R			$\cdots$		$\cdots$	
3	R			R		$\cdots$		$\cdots$	
$\vdots$	R				R	$\ldots$		$\cdots$	
x_1	R	?	?		?	$\cdots$	?	$\cdots$	?
$x_1 + 1$		?	?	?	?	$\cdots$	?	$\cdots$	?
$x_1 + 2$		?	?	?	?	$\cdots$	?	$\cdots$	?
$\vdots$		?	?	?	?	$\cdots$	?	$\cdots$	?
n		?	?	?	?	$\cdots$	?	$\cdots$	?

Table 1: The Grid in Three Parts

Part 1: The first column. This has x_1 elements of A in it.

Part 2: Consider the grid consisting of rows $1, \ldots, x_1$ and columns $2, \ldots, m$. Look at the j^{th} column, $2 \le j \le m$ in this grid. For each such j, this column has at most one element in A (else there would be a rectangle using the first column). Hence the total number of elements of A from this part of the grid is $m - 1$. (We drew them in a diagonal pattern though this is not required.)

Part 3: The bottom most $n - x_1$ elements of the right most $m - 1$ columns. This clearly has $\le \mathrm{maxrf}(n - x_1, m - 1)$ elements in it. We do not know which elements will be taken so we just use ?'s.

Taking all the parts into account we obtain

$$|A| \le x_1 + (m - 1) + \mathrm{maxrf}(n - x_1, m - 1).$$

∎

3 Tools for Finding c-colorings

3.1 Strong c-colorings and Strong (c, c')-colorings

Def 3.1 Let $c, c', n, m \in \mathbb{N}$ and let $\chi : G_{n,m} \to [c]$. Assume $c' \leq c$.

1. A *half-mono rectangle with respect to* χ is a rectangle where the left corners are the same color and the right corners are the same color.
2. χ is a *strong c-coloring* if there are no half-mono rectangles.
3. χ is a *strong* (c, c')*-coloring* if for any half-mono rectangle the color of the left corners and the right corners are (1) different, and (2) in $[c']$.

Example 3.2

1. Table 2 is a strong 4-coloring of $G_{5,8}$.

1	1	1	4	1	1	4	4
2	2	4	1	2	4	1	4
3	4	2	2	4	2	4	1
4	3	3	3	4	4	2	2
4	4	4	4	3	3	3	3

Table 2: Strong 4-coloring of $G_{5,8}$

2. Table 3 is a strong 3-coloring of $G_{4,6}$.

1	1	3	1	3	3
2	3	1	3	1	3
3	2	2	3	3	1
3	3	3	2	2	2

Table 3: Strong 3-coloring of $G_{4,6}$

3. Table 4 is a strong $(4, 2)$-coloring of $G_{6,15}$.
4. Table 5 is a strong $(6, 2)$-coloring of $G_{8,6}$.
5. Table 6 is a $(5, 3)$-coloring of $G_{8,28}$.

Theorem 3.3 *Let* $c, c', n, m \in \mathbb{N}$. *Let* $x = \lfloor c/c' \rfloor$. *If* $G_{n,m}$ *is strongly* (c, c')*-colorable then* $G_{n,xm}$ *is c-colorable.*

1	1	1	1	1	3	3	3	2	3	3	2	2	2	2
1	2	2	2	2	1	1	1	1	4	4	3	3	3	2
2	1	3	3	2	1	2	2	2	1	1	1	4	4	3
2	2	1	4	3	2	1	4	3	1	2	2	1	1	4
3	3	2	1	4	2	2	1	4	2	1	4	1	2	1
4	4	4	2	1	4	4	2	1	2	2	1	2	1	1

Table 4: Strong $(4, 2)$-coloring of $G_{6,15}$

1	1	2	2	3	6
1	2	1	2	4	5
2	1	2	1	5	4
2	2	1	1	6	3
3	4	5	6	1	2
4	5	6	4	1	1
5	6	3	3	1	2
6	3	4	5	1	2

Table 5: Strong $(6, 2)$-coloring of $G_{8,6}$

1	1	1	1	1	1	1	5	5	5	5	3	2	4	3	4	3	2	3	4	3	2	3	3	2	2	2	2
1	2	2	2	2	2	2	1	1	1	1	1	1	5	4	5	4	3	4	3	4	3	3	4	3	3	3	2
2	1	3	3	3	3	2	1	2	2	2	2	2	1	1	1	1	1	5	5	5	4	4	3	4	3	4	3
2	2	1	4	4	4	3	2	1	3	3	3	3	1	2	2	2	2	1	1	1	1	5	5	5	4	3	3
3	3	2	1	5	3	3	2	2	1	4	4	4	2	1	3	3	3	1	2	2	2	1	1	1	5	5	4
3	4	3	2	1	5	4	3	3	2	1	5	3	2	2	1	5	4	2	1	3	3	1	2	2	1	1	5
4	3	4	3	2	1	5	3	4	3	2	1	5	3	3	2	1	5	2	2	1	5	2	1	3	1	2	1
5	5	5	5	3	2	1	4	3	4	3	2	1	3	5	3	2	1	3	3	2	1	2	2	1	2	1	1

Table 6: Strong $(5, 3)$-coloring of $G_{8,28}$

Proof:

Let χ be a strong (c,c')-coloring of $G_{n,m}$. Let the colors be $\{1,\ldots,c\}$. Let χ^i be the coloring

$$\chi^i(a,b) = \chi(a,b) + i \pmod{c}.$$

(During calculations mod c we use $\{1,\ldots,c\}$ instead of the more traditional $\{0,\ldots,c-1\}$.)

Take $G_{n,m}$ with coloring χ. Place next to it $G_{n,m}$ with coloring $\chi^{c'}$. Then place next to that $G_{n,m}$ with coloring $\chi^{2c'}$ Keep doing this until you have $\chi^{(x-1)c'}$ placed. Table 7 is an example using the strong $(6,2)$-coloring of $G_{8,6}$ in Example 3.2.4 to obtain a 6-coloring of $G_{8,18}$. Since $c'=2$ and $x=3$ we will be shifting the colors first by 2 then by 4.

1	1	2	2	3	6	3	3	4	4	5	2	5	5	6	6	1	4
1	2	1	2	4	5	3	4	3	4	6	1	5	6	5	6	2	3
2	1	2	1	5	4	4	3	4	3	1	6	6	5	6	5	3	2
2	2	1	1	6	3	4	4	3	3	2	5	6	6	5	5	4	1
3	4	5	6	1	2	5	6	1	2	3	4	1	2	3	4	5	6
4	5	6	4	1	1	6	1	2	6	3	3	2	3	4	2	5	5
5	6	3	3	1	2	1	2	5	5	3	4	3	4	1	1	5	6
6	3	4	5	1	2	2	5	6	1	3	4	4	1	2	3	5	6

Table 7: Using the Strong $(6,2)$-coloring of $G_{8,6}$ to get a 6-coloring of $G_{8,18}$

We claim that the construction always creates a c-coloring of $G_{m,xn}$.

We show that there is no rectangle with the two leftmost points from the first $G_{n,m}$. From this, to show that there are no rectangles at all is just a matter of notation.

Assume that in column i_1 there are two points colored R (in this proof $1 \le R, B, G \le c$.) We call these *the i_1-points*. The points cannot form a rectangle with any other points in $G_{n,m}$ since χ is a c-coloring of $G_{n,m}$. The i_1-points cannot form a rectangle with points in columns i_1+m, i_1+2m, …, $i_1+(c-1)m$ since the colors of those points are $R+c' \pmod{c}$, $R+2c' \pmod{c}$, …, $R+(x-1)c' \pmod{c}$, all of which are not equal to R. Is there a j, $1 \le j \le x-1$ and a i_2, $1 \le i_2 \le m$ such that the i_1-points form a rectangle with points in column i_2+jm?

Since χ is a strong (c,c')-coloring, points in column i_2 and on the same row as the i_1-points are either colored *differently*, or both colors are in $[c']$. We consider both of these cases.

Case 1: In column i_2 the colors are B and G where $B \ne G$ (it is possible that $B=R$ or $G=R$ but not both). By the construction, the points in column i_2+jm are colored $B+jc'$ $\pmod{c}$ and $G+jc' \pmod{c}$. These points are colored differently, hence they cannot form a rectangle with the i_1-points.

$\cdots$	i_1	$\cdots$	i_2	$\cdots$	$\cdots$	i_1+jm	$\cdots$	i_2+jm	$\cdots$
$\cdots$	R	$\cdots$	B	$\cdots$	$\cdots$	$R+jc'$	$\cdots$	$B+jc'$	$\cdots$
$\cdots$	R	$\cdots$	G	$\cdots$	$\cdots$	$R+jc'$	$\cdots$	$G+jc'$	$\cdots$

Case 2: In column i_2 the colors are both B.

$$\begin{array}{ccccccc|ccccccc}
\cdots & i_1 & \cdots & i_2 & \cdots & & & \cdots & i_1+jm & \cdots & i_2+jm & \cdots \\
\hline
\cdots & R & \cdots & B & \cdots & & & \cdots & R+jc' & \cdots & B+jc' & \cdots \\
\cdots & R & \cdots & B & \cdots & & & \cdots & R+jc' & \cdots & B+jc' & \cdots
\end{array}$$

We have $R, B \in [c']$. By the construction, the points in column $i_2 + jm$ are both colored $B + jc' \pmod{c}$. We show that $R \not\equiv B + jc' \pmod{c}$. Since $1 \le j \le x-1$ we have

$$c' \le jc' \le (x-1)c'.$$

Hence

$$B + c' \le B + jc' \le B + (x-1)c'.$$

Since $B \in [c']$ we have $B + (x-1)c' \le xc'$. Hence

$$B + c' \le B + jc' \le xc'.$$

By the definition of x we have $xc' \le c$. Since $B \in [c']$ we have $B + c' \ge c' + 1$. Hence

$$c' + 1 \le B + jc' \le c.$$

Since $R \in [c']$ we have that $R \not\equiv B + jc'$. ∎

3.2 Using Combinatorics and Strong (c, c')-colorings

Theorem 3.4 *Let $c \ge 2$.*

1. *There is a strong c-coloring of $G_{c+1,\binom{c+1}{2}}$.*

2. *There is a c-coloring of $G_{c+1,m}$ where $m = c\binom{c+1}{2}$.*

Proof:
1) We first do an example of our construction. In the $c = 5$ case we obtain the coloring in Table 8

5	5	5	5	5	1	1	1	1	1	1	1	1	1	1
5	1	1	1	1	5	5	5	5	2	2	2	2	2	2
1	5	2	2	2	5	2	2	2	5	5	5	3	3	3
2	2	5	3	3	2	5	3	3	5	3	3	5	5	4
3	3	3	5	4	3	3	5	4	3	5	4	5	4	5
4	4	4	4	5	4	4	4	5	4	4	5	4	5	5

Table 8: Strong 5-coloring of $G_{6,15}$

Index the columns by $\binom{[c+1]}{2}$. Color rows of column $\{x, y\}$, $x < y$, as follows.

1. Color rows x and y with color c.

2. On the other spots use the colors $\{1, 2, 3, \ldots, c-1\}$ in increasing order (the actual order does not matter).

We call the coloring $\chi : G_{n,m} \to [c]$. We show that there are no half-mono rectangles. Let $RECT = \{p_1, p_2, q_1, q_2\}$ be a rectangle with p_1, p_2 in column $\{x, y\}$ and q_1, q_2 in column $\{x', y'\}$.

If any of p_1, p_2, q_1, q_2 have a color in $\{1, \ldots, c-1\}$ then $RECT$ cannot be a half-mono rectangle since the colors $\{1, \ldots, c-1\}$ only appear once in each column.

If $\chi(p_1) = \chi(p_2) = \chi(q_1) = \chi(q_2) = c$ then p_1 and p_2 are in rows x and y, and q_1 and q_2 are in rows x' and y'. Since $RECT$ is a rectangle $\{x, y\} = \{x', y'\}$. Hence p_1, p_2, q_1, q_2 are all in the same column. This contradicts $RECT$ being a rectangle.

2) This follows from Theorem 3.3 with $c = c$ and $c' = 1$, and Part (1) of this theorem. ∎

In order to generalize Theorem 3.4 we need a lemma. The lemma (and the examples) is based on the Wikipedia entry on Round Robin tournaments; hence we assume it is folklore. We present a proof for completeness.

Lemma 3.5 *Let $n \in \mathbb{N}$.*

1. *$\binom{[2n]}{2}$ can be partitioned into $2n-1$ sets $P_1, \ldots, P_{2n-1}$, each of size n, such that each P_i is itself a partition of $[2n]$ into pairs (i.e., a perfect matching) and all of the P_i's are disjoint.*

2. *For each $i \in [2n+1]$ $\binom{[2n+1]}{2}$ can be partitioned into $2n+1$ sets $P_1, \ldots, P_{2n+1}$, each of size n, such that each P_i is itself a partition of $[2n+1] - \{i\}$ into pairs (i.e., a perfect matching) and all of the P_i's are disjoint.*

Proof:
1) All arithmetic is mod $2n-1$ with two caveats: (a) we will use $\{1, 2, \ldots, 2n-1\}$ rather than the more traditional $\{0, 1, 2, \ldots, 2n-2\}$, (b) we will use the number $2n$ and not set it equal to 1; however, $2n$ will not be involved in any calculations. For $1 \le i \le 2n-1$ we have the following partition P_i:

$$\left| \begin{array}{c|c|c|c|c|c|c} 2n & i+1 & i+2 & \cdots & i+n-3 & i+n-2 & i+n-1 \\ i & i-1 & i-2 & \cdots & i-n+3 & i-n+2 & i-n+1 \end{array} \right|$$

Formally

$$P_i = \{2n, i\} \cup \{\{i+j, i-j\} : 1 \le j \le n-1\}.$$

It is easy to see that each P_i consists of disjoint pairs and that the P_i's are disjoint.
Example: $n = 4$. $1 \le i \le 7$.

P_1

$$\begin{vmatrix} 8 & 2 & 3 & 4 \\ 1 & 7 & 6 & 5 \end{vmatrix}$$

P_2

$$\begin{vmatrix} 8 & 3 & 4 & 5 \\ 2 & 1 & 7 & 6 \end{vmatrix}$$

P_3

$$\begin{vmatrix} 8 & 4 & 5 & 6 \\ 3 & 2 & 1 & 7 \end{vmatrix}$$

P_4

$$\begin{vmatrix} 8 & 5 & 6 & 7 \\ 4 & 3 & 2 & 1 \end{vmatrix}$$

P_5

$$\begin{vmatrix} 8 & 6 & 7 & 1 \\ 5 & 4 & 3 & 2 \end{vmatrix}$$

P_6

$$\begin{vmatrix} 8 & 7 & 1 & 2 \\ 6 & 5 & 4 & 3 \end{vmatrix}$$

P_7

$$\begin{vmatrix} 8 & 1 & 2 & 3 \\ 7 & 6 & 5 & 4 \end{vmatrix}$$

2) We partition $[2n+1]$. All arithmetic is be mod $2n+1$; however, we use $\{1, 2, \ldots, 2n+1\}$ rather than the more traditional $\{0, 1, 2, \ldots, 2n\}$. For $1 \le i \le 2n+1$ we have the following partition P_i:

$$\begin{vmatrix} i+1 & i+2 & i+3 & \cdots & i+n-3 & i+n-2 & i+n \\ i-1 & i-2 & i-3 & \cdots & i-n+3 & i-n+2 & i-n \end{vmatrix}$$

Formally

$$P_i = \{\{i+j, i-j\} : 1 \le j \le n\}\}.$$

It is easy to see that each P_i consists of disjoint pairs of $\{0, 1, \ldots, 2n\} - \{i\}$ and that the P_i's are disjoint.

Example: $n = 3$. $1 \le i \le 7$ and arithmetic is mod 7.

P_1

$$\begin{vmatrix} 2 & 3 & 4 \\ 7 & 6 & 5 \end{vmatrix}$$

P_2

$$\begin{vmatrix} 3 & 4 & 5 \\ 1 & 7 & 6 \end{vmatrix}$$

P_3

$$\begin{array}{|c|c|c|} 4 & 5 & 6 \\ 2 & 1 & 7 \end{array}$$

P_4

$$\begin{array}{|c|c|c|} 5 & 6 & 7 \\ 3 & 2 & 1 \end{array}$$

P_5

$$\begin{array}{|c|c|c|} 6 & 7 & 1 \\ 4 & 3 & 2 \end{array}$$

P_6

$$\begin{array}{|c|c|c|} 7 & 1 & 2 \\ 5 & 4 & 3 \end{array}$$

P_7

$$\begin{array}{|c|c|c|} 1 & 2 & 3 \\ 6 & 5 & 4 \end{array}$$

■

Theorem 3.6 *Let $c, c' \in \mathbb{N}$ with $c \geq 2$ and $1 \leq c' \leq c$.*

1. *There is a strong (c, c')-coloring of $G_{c+c',m}$ where $m = \binom{c+c'}{2}$.*
2. *There is a c-coloring of $G_{c+c',m'}$ where $m' = \lfloor c/c' \rfloor \binom{c+c'}{2}$.*

Proof:
1) We split into two cases.
Case 1: $c + c'$ is even. Then $c + c' = 2n$ for some n. Since $c' \leq c$, we also have $c' \leq n$. Let $P_1, \ldots, P_{2n-1}$ be the partition of $[2n]$ of Lemma 3.5.1. Index the elements of each P_i as $p_{i,j}$ for $1 \leq j \leq n$, that is, $P_i = \{p_{i,1}, p_{i,2}, \ldots, p_{i,n}\}$. We partition the $\binom{c+c'}{2}$ columns into $2n - 1$ parts of n columns each (note that $n(2n-1) = \binom{2n}{2}$). We color the j^{th} column in the i^{th} block as follows:

- The j^{th} column uses color 1 in the two rows row indexed by $p_{i,(j+1) \bmod n}$,
- The j^{th} column uses color 2 in the two rows row indexed by $p_{i,(j+2) \bmod n}$,
- $\vdots$
- The j^{th} column uses color c' in the two rows row indexed by $p_{i,(j+c') \bmod n}$,
- The j^{th} column uses the colors $c' + 1, \ldots, c$ once each to the rest of the elements in the column. For definiteness use them in increasing order.

We show that this yields a strong (c', c)-coloring. Assume there is a half-mono rectangle. Since every color in $\{c'+1, \ldots, c\}$ only appears once in a column we have that the left and right color are both in $[c']$. We need only prove that they are different. Assume, by way of contradiction, that the rectangle is monochromatic and colored d. Assume that one columns is column j_1 in part i_1 and the other is column j_2 in part i_2. It is possible that $i_1 = i_2$ or $j_1 = j_2$ but not both.

Subcase 1: $i_1 = i_2 = i$, so the two columns are in the same part. By the construction that $p_{i,(j_1+d) \bmod n} = p_{i,(j_2+d) \bmod n}$. Since all of the p_{ij}'s are different this means that $j_1 \equiv j_2 \pmod{n}$. Since $1 \le j_1, j_2 \le n$ we have $j_1 = j_2$.
Subcase 2: $i_1 \ne i_2$. By the construction this means that $p_{i_1,(j_1+d) \bmod n} = p_{i_2,(j_2+d) \bmod n}$. Since P_{i_1} and P_{i_2} are disjoint this cannot happen.

We now give some examples of colorings.
Example: $c' = 2$ and $c = 6$. $2n = c + c' = 8$ so $n = 4$. Note that $c + c' = 8$ and $\binom{c+c'}{2} = \binom{8}{2} = 28$. Our goal is to strongly 6-color $G_{8,28}$. We use the partitions $P_1, \ldots, P_7$ in the first example in Lemma 3.5. We first partition the 28 columns of $G_{7,28}$ into $2n - 1 = 7$ parts of $n = 4$ each:: $\{1, 2, 3, 4\}$, $\{5, 6, 7, 8\}$, $\{9, 10, 11, 12\}$, $\{13, 14, 15, 16\}$, $\{17, 18, 19, 20\}$, $\{21, 22, 23, 24\}$, $\{25, 26, 27, 28\}$.

We color the i^{th} $(1 \le i \le 9)$ set of columns using P_i to tell us where to put the 1's and 2's.

We describe the coloring of the first four columns carefully. The strong 6-coloring of $G_{8,28}$ is then in Table 9. then fill in the rest in a similar manner.

p_{11}	p_{12}	p_{13}	p_{14}
8	2	3	4
1	7	6	5

Fix $i = 1$, so we are looking at the 1^{st} part (the first four columns). Fix $j = 1$, so we are looking at the 1^{st} column of the 1^{st} part (the first column). We put a 1 in the rows indexed by $p_{i,j+1} = p_{1,2}$. So we put 1 in the 2^{st} and 7^{th} rows of the first column. We put a 2 in the rows indexed by $p_{i,j+2} = p_{1,3}$. So we put 2 in the 3^{st} and 6^{th} rows of the first column. The rest of the rows get $3, 4, 5, 6$ in increasing order.

Fix $j = 2$ (the second column of the first part, so the second column). We put a 1 in the rows indexed by $p_{i,j+1} = p_{1,3}$. So we put 1 in the 3^{st} and 6^{th} rows of the first column. We put a 2 in the rows indexed by $p_{i,j+2} = p_{1,4}$. So we put 2 in the 4^{st} and 5^{th} rows of the first column. The rest of the rows get $3, 4, 5, 6$ in increasing order.

Fix $j = 3$ (the third column of the first part, so the third column). We put a 1 in the rows indexed by $p_{i,j+1} = p_{1,4}$. So we put 1 in the 4^{st} and 5^{th} rows of the first column. We put a 2 in the rows indexed by $p_{i,j+2} = p_{1,1}$. So we put 2 in the 4^{st} and 5^{th} rows of the first column. The rest of the rows get $3, 4, 5, 6$ in increasing order.

Fix $j = 4$ (the fourth column of the first part, so the fourth column). We put a 1 in the rows indexed by $p_{i,j+1} = p_{1,1}$. So we put 1 in the 1^{st} and 8^{th} rows of the first column. We put a 2 in the rows indexed by $p_{i,j+2} = p_{1,2}$. So we put 2 in the 2[st] and 7[th] rows of the first column. The rest of the rows get $3, 4, 5, 6$ in increasing order.

3	3	2	1	1	3	3	2	2	1	3	3	3	2	1	3	3	2	1	3	2	1	3	3	1	3	3	2
1	4	3	2	3	4	2	1	1	3	4	2	2	1	3	4	4	2	1	4	3	2	1	4	2	1	4	3
2	1	4	3	1	5	4	2	3	4	2	1	1	3	4	2	2	1	3	5	4	2	1	5	3	2	1	4
4	2	1	4	2	1	5	3	1	5	5	2	4	4	2	1	1	3	4	2	2	1	4	6	4	2	1	5
5	2	1	5	4	2	1	4	2	1	6	4	1	5	5	2	5	4	2	1	1	3	5	2	2	1	5	6
2	1	5	6	5	2	1	5	4	2	1	5	2	1	6	5	1	5	5	2	5	4	2	1	1	4	6	2
1	5	6	2	6	6	6	6	5	2	1	6	5	2	1	6	2	1	6	6	1	5	6	2	5	5	2	1
6	6	2	1	2	1	2	1	6	6	2	1	6	6	2	1	6	6	2	1	6	6	2	1	6	6	2	1

Table 9: Strong 6-coloring of $G_{8,28}$.

Case 2: $c + c'$ **is odd.** Let $c + c' = 2n + 1$ for some n. Since $c' < c$, we also have $c' \leq n$. Let $P_1, \ldots, P_{2n+1}$ be from Lemma 3.5.2. Index the elements of each P_i as $p_{i,j}$ for $1 \leq j \leq n$, that is, $P_i = \{p_{i,1}, p_{i,2}, \ldots, p_{i,n}\}$. We partition the $\binom{c+c'}{2}$ columns into $2n + 1$ parts of n columns each (note that $n(2n + 1) = \binom{2n+1}{2}$). The description of the coloring and the proof that it works are identical to that in Case 1, hence we omit it.

Example: $c' = 3$ and $c = 4$. $2n + 1 = c + c' = 7$ so $n = 3$. Note that $c + c' = 7$ and $\binom{c+c'}{2} = \binom{7}{2} = 21$. Our goal is to strongly 5-color $G_{7,21}$. We use the partitions $P_1, \ldots, P_7$ in the second example in Lemma 3.5. We first partition the 21 columns of $G_{7,21}$ into $2n+1 = 7$ parts of $n = 3$ each:: $\{1, 2, 3\}$, $\{4, 5, 6\}$, $\{7, 8, 9\}$, $\{10, 11, 12\}$, $\{13, 14, 15\}$, $\{16, 17, 18\}$, $\{19, 20, 21\}$

We color the i^{th} $(1 \leq i \leq 7)$ set of columns using P_i to tell us where to put the 1's and 2's. The final coloring is in Table 10.

3	3	3	3	2	1	1	3	2	2	1	3	2	1	3	1	3	2	3	2	1
4	2	1	4	3	3	3	2	1	1	3	2	2	1	4	2	1	3	1	3	2
1	4	2	5	2	1	4	4	3	3	2	1	1	3	2	2	1	4	2	1	3
2	1	4	1	4	2	5	2	1	4	4	4	3	2	1	1	4	2	2	1	4
2	1	5	2	1	4	1	5	2	5	2	1	4	4	5	3	2	1	1	4	2
1	5	2	2	1	5	2	1	4	1	5	2	5	2	1	4	5	5	4	2	2
5	2	1	1	5	2	2	1	5	2	1	5	1	5	2	5	2	1	6	5	5

Table 10: Strong 5-coloring of $G_{7,21}$.

2) This follows from Theorem 3.3 and Part (1) of this theorem. ∎

Corollary 3.7 *For all $c \geq 2$, there is a c-coloring of $G_{2c,2c^2-c}$.*

3.3 Using Finite Fields and Strong c-colorings

Def 3.8 Let X be a finite set and $q \in \mathbb{N}$, $q \geq 3$. Let $P \subseteq \binom{X}{q}$.

$$\text{pairs}(P) = \{\{a_1, a_2\} \in \binom{X}{2} : (\exists a_3, \ldots, a_q)[\{a_1, \ldots, a_q\} \in P]\}.$$

Example 3.9 Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let $q = 3$.

1. Let $P = \{\{1, 2, 6\}, \{1, 8, 9\}, \{2, 4, 6\}\}$. Then
$$\text{pairs}(P) = \{\{1, 2\}, \{1, 6\}, \{2, 6\}, \{1, 8\}, \{1, 9\}, \{8, 9\}, \{2, 4\}, \{4, 6\}\}$$
2. Let $P = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}\}$. Then
$$\text{pairs}(P) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4, 5\}, \{4, 6\}, \{5, 6\}, \{7, 8\}, \{7, 9\}, \{8, 9\}\}.$$

Theorem 3.10 *Let $c, m, r \in \mathbb{N}$. Assume there exist $P_1, \ldots, P_m \subseteq \binom{[cr]}{r}$ such that the following hold.*

- *For all $1 \leq j \leq m$, P_j is a partition of $[cr]$ into c parts of size r.*
- *For all $1 \leq j_1 < j_2 \leq m$, $\text{pairs}(P_{j_1}) \cap \text{pairs}(P_{j_2}) = \emptyset$.*

Then

1. *$G_{cr,m}$ is strongly c-colorable.*
2. *$G_{cr,cm}$ is c-colorable.*

Proof:
1)

We define a strong c-coloring COL of $G_{cr,m}$ using $P_1, \ldots, P_m$.
Let $1 \leq j \leq m$. Let
$$P_j = \{L_j^1, \ldots, L_j^c\}$$
where each L_j^i is a subset of r elements from $[cr]$.

Let $1 \leq i \leq cr$ and $1 \leq j \leq m$. Since P_j is a partition of $[cr]$ there exists a unique u such that $i \in L_j^u$. Define
$$COL(i, j) = u.$$

We show that this is a strong c-coloring. Assume, by way of contradiction, that there exists $1 \leq i_1 < i_2 \leq 2k$ and $1 \leq j_1 < j_2 \leq 2k - 1$ such that $COL(i_1, j_1) = COL(i_1, j_2) = u$ and $COL(i_2, j_1) = COL(i_2, j_2) = v$. By definition of the coloring we have

$$i_1 \in L_{j_1}^u, i_1 \in L_{j_2}^u, i_2 \in L_{j_1}^v, i_2 \in L_{j_2}^v$$

Then

$$\{i_1, i_2\} \in \text{pairs}(P_{j_1}) \cap \text{pairs}(P_{j_2}),$$

contradicting the second premise on the P's.
2) This follows from Part (1) and Theorem 3.3 with $c = c$ and $c' = 1$. ∎

The Round Robin partition of Lemma 3.5 is an example of a partition satisfying the premises of Theorem 3.10, where $c = n$, $r = 2$, and $m = 2n - 1 = 2c - 1$. The next theorem yields partitions with bigger values of r.

Theorem 3.11 *Let p be a prime and $s, d \in \mathbb{N}$.*

1. *$G_{p^{ds}, \frac{p^{ds}-1}{p-1}}$ is strongly p^{ds-s}-colorable.*
2. *$G_{p^{ds}, \frac{p^{ds}-1}{p-1}p^{ds-s}}$ is p^{ds-s}-colorable.*

Proof: Let $c = p^{ds-s}$, $r = p^s$, and $m = \frac{p^{ds}-1}{p^s-1}$. We show that there exists $P_1, \ldots, P_m$ satisfying the premise of Theorem 3.10. The result follows immediately.

Let F be the finite field on p^s elements. We identify $[cr]$ with the set F^d.

Def 3.12

1. Let $\vec{x} \in F^d$, $\vec{y} \in F^d - \{0^d\}$. Then

$$L_{\vec{x},\vec{y}} = \{\vec{x} + f\vec{y} \mid f \in F\}.$$

   Sets of this form are called *lines*. Note that for all $\vec{x}, \vec{y}, a \in F$ with $a \neq 0$,

$$L_{\vec{x},\vec{y}} = L_{\vec{x},a\vec{y}}.$$

2. Two lines $L_{\vec{x},\vec{y}}$, $L_{\vec{z},\vec{w}}$ *have the same slope* if $\vec{y}$ is a multiple of $\vec{w}$.

The following are easy to prove and well-known.

- If L and L' are two distinct lines that have the same slope, then $L \cap L' = \emptyset$.
- If L and L' are two distinct lines with different slopes, then $|L \cap L'| \leq 1$.
- If L is a line then there are exactly $r = p^s$ points on L.
- If L is a line then there are exactly $c = p^{ds-s}$ lines that have the same slope as L (this includes L itself).
- There are exactly $\frac{p^{ds}-1}{p^s-1}$ different slopes.

We define $P_1, \ldots, P_m$ as follows.

1. Pick a line L. Let P_1 be the set of lines that have the same slope as L.

2. Assume that $P_1, \ldots, P_{j-1}$ have been defined and that $j \le m$. Let L be a line that is not in $P_1 \cup \cdots \cup P_{j-1}$. Let P_j be the set of all lines that have the same slope as L.

We need to show that $P_1, \ldots, P_m$ satisfies the premises of Theorem 3.10

a) For all $1 \le j \le m$, P_j is a partition of $[cr]$ into c parts of size r. Let $L \in P_j$. Note that P_j is the set of all lines with the same slope as L. Clearly this partitions F^d which is $[cr]$.

b) For all $1 \le j_1 < j_2 \le m$, $\text{pairs}(P_{j_1}) \cap \text{pairs}(P_{j_2}) = \emptyset$. Let L_1 be any line in P_{j_1} and L_2 be any line in P_{j_2}. Since $|L_1 \cap L_2| \le 1 < 2$ we have the result.

Note that each P_j has $c = p^{ds-s}$ sets (lines) in it, each set (line) has $r = p^s$ numbers (points), and there are $m = \frac{p^{ds}-1}{p^s-1}$ many P's. Hence the premises of Theorem 3.10 are satisfied. ▮

It is convenient to state the $s = 1$, $d = 2$ case of Theorem 3.11.

Corollary 3.13 *Let p be a prime.*

1. *There is a strong p-coloring of $G_{p^2,p+1}$.*

2. *There is a p-coloring of G_{p^2,p^2+p}.*

3.4 Using Finite Fields for the Square and Almost Square Case

Can Theorem 3.11 be used to get that, if c is a prime power, G_{c^2,c^2} is c-colorable. Not quite. If $d = 2$ one obtains that a grid of dimensions $p^{2s} \times \frac{p^s-1}{p-1}p^s$ is p^d-colorable. Letting $c = p^s$ one gets that if c is a prime power then $c^2 \times c^{2-(1/s)+o(1)}$ is c-colorable.

Ken Berg and Quimey Vivas have both shown (independently) that if c is a prime power then G_{c^2,c^2} is c-colorable. (They both emailed us their proofs.) Ken Berg extended this to show that if c is a prime power then G_{c^2,c^2+c} is c-colorable. We present both proofs. This result is orthogonal to Theorem 3.11 in that there are results you can get from either that you cannot get from the other.

Theorem 3.14 *If c is a prime power then G_{c^2,c^2} is c-colorable.*

Proof:

Let F be a field of c elements. We view the elements of G_{c^2,c^2} as indexed by $(F \times F) \times (F \times F)$. The colorings is

$$COL((x_1, x_2), (y_1, y_2)) = x_1 y_1 + x_2 + y_2.$$

Note that all of this arithmetic takes place in the field F.

Assume, by way of contradiction, that there is a monochromatic rectangle. Then there exists $w_1, w_2, x_1, x_2, y_1, y_2, z_1, z_2 \in F$ such that $((w_1, w_2), (x_1, x_2))$, $((w_1, w_2), (y_1, y_2))$, $((z_1, z_2), (x_1, x_2))$, and $((z_1, z_2), (y_1, y_2))$ are all distinct and

$COL((w_1,w_2),(x_1,x_2)) = COL((w_1,w_2),(y_1,y_2)) = COL((z_1,z_2),(x_1,x_2)) = COL((z_1,z_2),(y_1,y_2))$.

Since $COL((w_1,w_2),(x_1,x_2)) = COL((w_1,w_2),(y_1,y_2))$

$$\begin{aligned} w_1x_1 + w_2 + x_2 &= w_1y_1 + w_2 + y_2 \\ w_1(x_1 - y_1) &= y_2 - x_2 \end{aligned}$$

Since $COL((z_1,z_2),(x_1,x_2)) = COL((z_1,z_2),(y_1,y_2))$

$$\begin{aligned} z_1x_1 + z_2 + x_2 &= z_1y_1 + z_2 + y_2 \\ z_1(x_1 - y_1) &= y_2 - x_2 \end{aligned}$$

Combining these two we get

$$\begin{aligned} w_1(x_1 - y_1) &= z_1(x_1 - y_1) \\ (w_1 - z_1)(x_1 - y_1) &= 0 \end{aligned}$$

Since the arithmetic takes place in a field we obtain that either $w_1 = z_1$ or $x_1 = y_1$.

Case 1: $w_1 = z_1$

Since $COL((w_1,w_2),(x_1,x_2)) = COL((z_1,z_2),(x_1,x_2))$.

$$\begin{aligned} w_1x_1 + w_2 + x_2 &= z_1x_1 + z_2 + x_2 \\ z_1x_1 + w_2 + x_2 &= z_1x_1 + z_2 + x_2 \text{ Since } w_1 = z_1. \\ w_2 &= z_2 \end{aligned}$$

Since $w_1 = z_1$ and $w_2 = z_2$ the four points are not distinct. This is a contradiction.

Case 2: $x_1 = y_1$. Similar to Case 1.

∎

Theorem 3.15 *If c is a prime power then G_{c^2,c^2+c} is c-colorable.*

Proof:

Let F be a field of c elements. Let $*$ be a symbol to which we assign no meaning. We view the elements of G_{c^2,c^2+c} as indexed by $(F \times F) \times (F \cup \{*\} \times F)$.

We describe the coloring. Assume $x_1, x_2, y_1, y_2 \in F$.

$$\begin{aligned} COL((x_1,x_2),(y_1,y_2)) &= x_1y_1 + x_2 + y_2 \\ COL((x_1,x_2),(*,y_2)) &= x_1 + y_2 \end{aligned}$$

Note that all of this arithmetic takes place in the field F.

Assume, by way of contradiction, that there is a monochromatic rectangle. Then there exists $w_1, w_2, x_1, x_2, y_1, y_2, z_1, z_2$ such that

- $w_1, w_2, x_2, y_2, z_1, z_2 \in F$
- $x_1, y_1 \in F \cup \{*\}$.

- $((w_1,w_2),(x_1,x_2))$, $((w_1,w_2),(y_1,y_2))$, $((z_1,z_2),(x_1,x_2))$, $((z_1,z_2),(y_1,y_2))$ are all distinct.
- $((w_1,w_2),(x_1,x_2))$, $((w_1,w_2),(y_1,y_2))$, $((z_1,z_2),(x_1,x_2))$, $((z_1,z_2),(y_1,y_2))$ are all the same color.

By the proof of Theorem 3.14 at least one of x_1, y_1 is $*$. We can assume $x_1 = *$. There are two cases.
Case 1: $y_1 = *$. Since

$$COL((w_1,w_2),(*,x_2)) = COL((w_1,w_2),(*,y_2))$$

we have

$$w_1 + x_2 = w_1 + y_2$$

so $x_2 = y_2$. Hence $(x_1,x_2) = (y_1,y_2)$ so the points are not distinct.
Case 2: $y_1 \neq *$. Since

$$COL((w_1,w_2),(*,x_2)) = COL((z_1,z_2),(*,x_2))$$

we have

$$w_1 + x_2 = z_1 + x_2$$

so $w_1 = z_1$. Since

$$COL((w_1,w_2),(y_1,y_2)) = COL((z_1,z_2),(y_1,y_2))$$

we have

$$w_1y_1 + w_2 + y_2 = z_1y_1 + z_2 + y_2.$$

Since $w_1 = z_1$ we have $w_2 = z_2$. Hence we have $(w_1,w_2) = (z_1,z_2)$ so the points are not distinct.
∎

4 Bounds on the Sizes of Obstruction Sets

4.1 An Upper Bound

Using the uncolorability bounds, we can obtain an upper-bound on the size of a c-colorable grid.

Theorem 4.1 *For all $c > 0$, G_{c^2+c,c^2+c} is not c-colorable.*

Proof: We apply Corollary 2.9 with $m = c^2 + c$ and $n = c^2 + c$. Note that

$$\begin{aligned} \left\lceil \frac{nm}{c} \right\rceil &= \left\lceil \frac{(c^2+c)(c^2+c)}{c} \right\rceil \\ &= (c+1)(c^2+c). \end{aligned}$$

Letting $q = c + 1$ and $r = 0$, we have

$$\begin{aligned} \frac{m(m-1) - 2qr}{q(q-1)} &= \frac{(c^2+c)(c^2+c-1)}{(c+1)c} \\ &= c^2 + c - 1 \\ &< c^2 + c \\ &= n. \end{aligned}$$

▮

Using this, we can obtain an upper-bound on the size of an obstruction set.

Theorem 4.2 *If $c > 0$, then $|\mathrm{OBS}_c| \leq 2c^2$.*

Proof: For each r, there can be at most one element of OBS_c of the form $G_{r,n}$. Likewise, there can be at most one element of OBS_c of the form $G_{n,r}$. If $r \leq c$ then for all n, $G_{r,n}$ and $G_{n,r}$ are trivially c-colorable and are, therefore, not an element of OBS_c. Theorem 4.1 shows that for all $n, m > c^2 + c$, $G_{n,m}$ is not an element of OBS_c. It follows that there can be at most two elements of OBS_c for each integer r where $c < r \leq c^2 + c$. Therefore $|\mathrm{OBS}_c| \leq 2c^2$. ▮

Note 4.3 We will later see that $|OBS_2| = 3$, $|OBS_3| = 8$, and $|OBS_4| = 16$. Based on this (scant) evidence the bound of $2c^2$ looks like its too large.

4.2 A Lower Bound

To get a lower bound on $|\mathrm{OBS}_c|$, we will combine Corollary 2.12 and Theorem 3.6(2) with the following lemma:

Lemma 4.4 *Suppose that $G_{m_1,n}$ is c-colorable and $G_{m_2,n}$ is not c-colorable. Then there exists n, m such that $m_1 < x \leq m_2$, $y \leq n$, and a grid $G_{x,y} \in \mathrm{OBS}_c$.*

Proof: Given n, let x be the least integer such that $G_{x,n}$ is not c-colorable. Clearly, $m_1 < x \leq m_2$. Now given x as above, let y be least such that $G_{x,y}$ is not c-colorable. Clearly, $y \leq n$ and $G_{x,y} \in \mathrm{OBS}_c$. ▮

Theorem 4.5 $|\mathrm{OBS}_c| \geq 2\sqrt{c}(1 - o(1))$.

Proof: For any $c \geq 2$ and any $1 \leq c' \leq c$ we can summarize Corollary 2.12 and Theorem 3.6(2) as follows:

$$G_{c+c',n} \text{ is } \begin{cases} c\text{-colorable} & \text{if } n \leq \left\lfloor \frac{c}{c'} \right\rfloor \binom{c+c'}{2}, \\ \text{not } c\text{-colorable} & \text{if } n > \frac{c}{c'} \binom{c+c'}{2}. \end{cases}$$

(We won't use the fact here, but note that this is tight if c' divides c.)

Suppose $c' > 1$ and

$$\frac{c}{c'} \binom{c+c'}{2} < \left\lfloor \frac{c}{c'-1} \right\rfloor \binom{c+c'-1}{2}. \tag{1}$$

Then letting $n := \left\lfloor \frac{c}{c'-1} \right\rfloor \binom{c+c'-1}{2}$, we see that $G_{c+c'-1,n}$ is c-colorable, but $G_{c+c',n}$ is not. Then by Lemma 4.4, there is a grid $G_{c+c',y} \in \mathrm{OBS}_c$ for some y. So there are at least as many elements of OBS_c as there are values of c' satisfying Inequality (1)—actually twice as many, because $G_{n,m} \in \mathrm{OBS}_c$ iff $G_{m,n} \in OBS_c$.

Fix any real $\varepsilon > 0$. Clearly, Inequality (1) holds provided

$$\frac{c}{c'} \binom{c+c'}{2} \leq \left(\frac{c}{c'-1} - 1 \right) \binom{c+c'-1}{2}.$$

A rather tedious calculation reveals that if $2 \leq c' \leq (1-\varepsilon)\sqrt{c}$, then this latter inequality holds for all large enough c. Including the grid $G_{c+1,n} \in \mathrm{OBS}_c$ where $n = c\binom{c+1}{2} + 1$, we then get $|\mathrm{OBS}_c| \geq \lfloor (1-\varepsilon)\sqrt{c} \rfloor$ for all large enough c, and since ε was arbitrary, we therefore have $|\mathrm{OBS}_c| \geq \sqrt{c}(1 - o(1))$.

To double the count, we notice that $c + c' \leq \left\lfloor \frac{c}{c'} \right\rfloor \binom{c+c'}{2}$, hence $G_{c+c',c+c'}$ is c-colorable by Theorem 3.6(2). This means that $G_{c+c',y} \in \mathrm{OBS}_c$ for some $y > c + c'$, and so we can count $G_{y,c+c'} \in \mathrm{OBS}_c$ as well without counting any grids twice. ∎

5 Which Grids Can be 2-Colored?

Theorem 5.1

1. *$G_{7,3}$ and $G_{3,7}$ are not 2-colorable*
2. *$G_{5,5}$ is not 2-colorable.*
3. *$G_{7,2}$ and $G_{2,7}$ are 2-colorable (this is trivial).*
4. *$G_{6,4}$ and $G_{4,6}$ are 2-colorable.*

Proof:

We only consider grids of the form $G_{n,m}$ where $n \geq m$.

1,2)

In Table 11 we show that $G_{7,3}$ and $G_{5,5}$ are not 2-colorable. For each (n,m) we use either Corollary 2.8 or 2.9. In the table we give, for each (n,m), the value of $\lceil \frac{nm}{2} \rceil$, the q, r such that $\lceil \frac{nm}{2} \rceil = qn + r$ with $0 \leq r \leq n-1$, which corollary we use (Use), the premise of the corollary (Prem), and the arithmetic showing the premise is true (Arith).

m	n	$\lceil \frac{nm}{2} \rceil$	q	r	Use	Prem	Arith
3	7	11	1	4	Cor 2.8	$\frac{m(m-1)}{2} < r \leq n$	$3 < 4 \leq 7$
5	5	13	2	3	Cor 2.9	$\frac{m(m-1)-2qr}{q(q-1)} < n$	$4 < 5$

Table 11: (m,n) such that $G_{m,n}$ is not 2-colorable

3) $G_{7,2}$ is clearly 2-colorable.

4) $G_{6,4}$ is 2-colorable by Corollary 3.13 with $p = 2$. We present that coloring in Table 12 below.

R	R	R	B	B	B
R	B	B	B	R	R
B	R	B	R	B	R
B	B	R	R	R	B

Table 12: 2-Coloring of $G_{4,6}$

▮

Theorem 5.2 $\mathrm{OBS}_2 = \{G_{7,3}, G_{5,5}, G_{3,7}\}$.

Proof:

$G_{7,3}$ is not 2-colorable by Theorem 5.1. $G_{6,3}$ is 2-colorable by Theorem 5.1. $G_{7,2}$ is 2-colorable by Theorem 5.1. Hence $G_{7,3} \in \mathrm{OBS}_2$. The proof for $G_{3,7}$ is similar.

$G_{5,5}$ is not 2-colorable by Theorem 5.1. $G_{5,4}$ and $G_{4,5}$ are 2-colorable by Theorem 5.1. Hence $G_{5,5} \in OBS_2$.

Table 13 indicates exactly which grids are 2-colorable. The entry for (n,m) is C if $G_{n,m}$ is 2-colorable, and N if $G_{n,m}$ is not 2-colorable. From this Table one easily sees that the grids listed in this theorem are the only elements of OBS_2.

▮

	2	3	4	5	6	7	8
2	C	C	C	C	C	C	C
3	C	C	C	C	C	N	N
4	C	C	C	C	C	N	N
5	C	C	C	N	N	N	N
6	C	C	C	N	N	N	N
7	C	N	N	N	N	N	N
8	C	N	N	N	N	N	N

Table 13: 2-Colorable Grids (C) and non 2-Colorable Grids (N)

6 Which Grids Can be 3-Colored?

Theorem 6.1

1. *$G_{19,4}$ and $G_{4,19}$ are not 3-colorable.*
2. *$G_{16,5}$ and $G_{5,16}$ are not 3-colorable.*
3. *$G_{13,7}$ and $G_{7,13}$ are not 3-colorable.*
4. *$G_{12,10}$ and $G_{10,12}$ are not 3-colorable.*
5. *$G_{11,11}$ is not 3-colorable.*
6. *$G_{19,3}$ and $G_{3,19}$ are 3-colorable (this is trivial).*
7. *$G_{18,4}$ and $G_{4,18}$ are 3-colorable.*
8. *$G_{15,6}$ and $G_{6,15}$ are 3-colorable.*
9. *$G_{12,9}$ and $G_{9,12}$ are 3-colorable.*

Proof:

We just consider the grids $G_{n,m}$ were $n \geq m$.

$1, 2, 3, 4, 5)$

In Table 14 we show that several grids are not 3-colorable. For each (n, m) we use either Corollary 2.8 or 2.9. In the table we give, for each (n, m), the value of $\lceil \frac{nm}{3} \rceil$, the q, r such that $\lceil \frac{nm}{3} \rceil = qn + r$ with $0 \leq r \leq n - 1$, which corollary we use (Use), the premise of the corollary (Prem), and the arithmetic showing the premise is true (Arith).

6) $G_{19,3}$ is clearly 3-colorable.

7) $G_{18,4}$ is 3-colorable by Theorem 3.4 with $c = 3$.

8) $G_{15,6}$ is 3-colorable by Corollary 3.7 with $c = 3$.

m	n	$\lceil \frac{nm}{3} \rceil$	q	r	Use	Prem	Arith
4	19	26	1	7	Cor 2.8	$\frac{m(m-1)}{2} < r \le n$	$6 < 7 \le 19$
5	16	27	1	11	Cor 2.8	$\frac{m(m-1)}{2} < r \le n$	$10 < 11 \le 16$
7	13	31	2	5	Cor 2.9	$\frac{m(m-1)-2qr}{q(q-1)} < n$	$11 < 13$
10	12	40	3	4	Cor 2.9	$\frac{m(m-1)-2qr}{q(q-1)} < n$	$11 < 12$
11	11	41	3	8	Cor 2.9	$\frac{m(m-1)-2qr}{q(q-1)} < n$	$10\frac{1}{3} < 11$

Table 14: (m, n) such that $G_{m,n}$ is not 3-colorable

9) $G_{12,9}$ is 3-colorable by Corollary 3.13 with $p = 3$. ∎

Theorem 6.2 *$G_{10,10}$ is 3-colorable.*

Proof: The 3-coloring is in Table 15.

1	1	1	1	2	2	3	3	2	3
1	2	2	3	1	1	1	3	3	2
3	1	2	3	1	2	2	1	1	3
3	2	1	2	2	1	3	1	3	1
1	2	3	3	3	2	3	2	1	1
3	1	2	2	3	3	1	2	2	1
2	3	1	2	3	2	1	3	1	2
2	2	3	1	1	3	2	3	2	1
3	3	3	1	2	1	2	2	1	2
2	3	2	1	2	3	1	1	3	3

Table 15: 3-Coloring of $G_{10,10}$.

∎

Note 6.3 We found the coloring in Theorem 6.2 by the following steps.

- We found a size 34 rectangle free subset of $G_{10,10}$ (by hand). Frankly we were trying to prove there was no such rectangle free set and hence $G_{10,10}$ would not be 3-colorable.
- We used the rectangle free set for color 1 and completed the coloring with a simple computer program.

It is an open problem to find a general theorem that has a corollary that $G_{10,10}$ is 3-colorable.

Theorem 6.4 *If $A \subseteq G_{11,10}$ and A is rectangle-free then $|A| \le 36 = \lceil \frac{11 \cdot 10}{3} \rceil - 1$. Hence $G_{11,10}$ is not 3-colorable.*

Proof:

We divide the proof into cases. Every case will either conclude that $|A| \le 36$ or A cannot exist.

For $1 \le j \le 10$ let x_j be the number of elements of A in column j. We assume

$$x_1 \ge \cdots \ge x_{10}.$$

1. $5 \le x_1 \le 11$.

   By Lemma 2.16 with $x = 5$, $n = 11$, $m = 10$ we have

   $$|A| \le x+m-1+\text{maxrf}(n-x, m-1) \le 5+10-1+\text{maxrf}(11-5, 10-1) \le 14+\text{maxrf}(6,9).$$

   By Lemma 11.1 we have $\text{maxrf}(6,9) = 21$. Hence

   $$|A| \le 14 + 21 = 35 \le 36.$$

2. There exists k, $0 \le k \le 6$, such that $x_1 = \cdots = x_k = 4$ and $x_{k+1} \le 3$. Then

   $$|A| = \sum_{j=1}^{10} x_j = (\sum_{j=1}^{k} x_j) + (\sum_{j=k+1}^{10} x_j) \le 4k + 3(10-k) = 30 + k$$

   Since $k \le 6$ this quantity is $\le 30 + 6 = 36$. Hence $|A| \le 36$.

3. $x_1 = \cdots = x_7 = 4$. Let G' be the grid restricted to the first 7 columns. Let B be A restricted to G'.

   (a) There exists $1 \le j_1 < j_2 < j_3 \le 7$ such that

   $$|C_{j_1} \cap C_{j_2} \cap C_{j_3}| = 1.$$

   By renumbering we can assume that

   $$|C_1 \cap C_2 \cap C_3| = 1$$

   and that the intersection is in row 10. The following picture summarizes our knowledge. We use R to denote where an element of A is.

	1	2	3	4	5	6	7
1	R						
2	R						
3	R						
4		R					
5		R					
6		R					
7			R				
8			R				
9			R				
10	R	R	R				
11							

Let $4 \leq j \leq 7$. In column j there can be at most 1 R in rows 1,2,3, at most 1 R in rows 4,5,6, at most 1 R in row 7,8,9,10. Hence, since $x_j = 4$, the jth column has an R in the 11th row. Also note that there must be exactly 1 R among rows 1,2,3, exactly 1 R among rows 4,5,6, and exactly one R among rows 7,8,9,10.

One can easily show that after a permutation of the rows we must have the following in the first 6 columns:

	1	2	3	4	5	6	7
1	R			R			
2	R				R		
3	R					R	
4		R		R			
5		R			R		
6		R				R	
7			R	R			
8			R		R		
9			R			R	
10	R	R	R				
11				R	R	R	R

It is easy to see that if an R is placed anywhere in column 7 then a rectangle is formed.

(b) There exists $1 \leq j_1 < j_2 < j_3 \leq 7$ such that $|C_{j_1} \cap C_{j_2}| = |C_{j_1} \cap C_{j_2}| = |C_{j_2} \cap C_{j_3}| = 1$. We can assume that for all sets of three columns their intersection is $\emptyset$ (else we would be in Case a). We can assume that $j_1 = 1$, $j_2 = 2$, $j_3 = 3$ and that the first three columns are as in the picture below.

	1	2	3	4	5	6	7
1	R						
2	R						
3		R					
4		R					
5			R				
6			R				
7	R	R					
8	R		R				
9		R	R				
10							
11							

Since no three columns intersect there can be no R's in the 7th, 8th, or 9th row of columns 4,5,6,7. In later pictures we will use X to denote that an R cannot be in that space.

There are essentially five ways that the columns 4,5,6,7 and rows 10,11 can be arranged. Here are all of them:

	4	5	6	7
10				
11	R	R	R	R

	4	5	6	7
10				R
11	R	R	R	

	4	5	6	7
10			R	R
11	R	R		

	4	5	6	7
10				R
11	R	R	R	R

	4	5	6	7
10			R	R
11	R	R	R	

The third one is the hardest to analyze. Hence we analyze that one and leave the rest to the reader. We can assume the following picture happens.

	1	2	3	4	5	6	7
1	R						
2	R						
3		R					
4		R					
5			R				
6			R				
7	R	R		X	X	X	X
8	R		R	X	X	X	X
9		R	R	X	X	X	X
10						R	R
11				R	R		

In columns 4,5,6,7 there must be exactly one R from row 1 or 2, exactly one R from row 3 or 4, and exactly one R from row 5 or 6. If we look just at columns 4 and 5, and permute the rows as needed, we can assume that the following picture happens:

	1	2	3	4	5	6	7
1	R			R			
2	R				R		
3		R		R			
4		R			R		
5			R	R			
6			R		R		
7	R	R		X	X	X	X
8	R		R	X	X	X	X
9		R	R	X	X	X	X
10						R	R
11				R	R		

Either there is an R at both (Row1,Col6) and (Row2,Col7) or there is an R at both (Row1,Col7) and (Row2,Col6). We call the first one *slanting NW-SE* and the former *slanting NE-SW*. Similar conditions apply for Rows 3 and 4, and Rows 5 and 6. Two of the pairs of rows must have the same slant. We can assume that Rows 1,2 and Rows 3,4 both slant NW-SE (the other cases are similar). We can assume that the following picture happens:

	1	2	3	4	5	6	7
1	R			R		R	
2	R				R		R
3		R		R		R	
4		R			R		R
5			R	R			
6			R		R		
7	R	R		X	X	X	X
8	R		R	X	X	X	X
9		R	R	X	X	X	X
10						R	R
11				R	R		

Clearly a rectangle is formed.

(c) There exists $1 \le j_1 < j_2 < j_3 \le 7$ such that $|C_{j_1} \cap C_{j_2}| = |C_{j_1} \cap C_{j_3}| = 1$ but $|C_{j_2} \cap C_{j_3}| = 0$. We can assume that for all sets of three columns their intersection is $\emptyset$ (else we would be in Case a). We can assume that $j_1 = 1$, $j_2 = 2$, $j_3 = 3$ and that the first three columns are as in the picture below.

	1	2	3	4	5	6	7
1	R						
2	R						
3	R	R					
4	R		R				
5		R					
6		R					
7		R					
8			R				
9			R				
10			R				
11							

The proof is similar to the proof of case 3a.

(d) There exists $1 \le j_1 < j_2 < j_3 \le 7$ such that $|C_{j_1} \cap C_{j_2}| = |C_{j_1} \cap C_{j_3}| = |C_{j_1} \cap C_{j_3}| = 0$. We can assume that $C_{j_1} = C_1$ and has rows $1, 2, 3, 4$, $C_{j_2} = C_2$ and has rows $5, 6, 7, 8$, and $C_{j_3} = C_3$ and has rows $9, 10, 11, 12$. Too bad we only have 11 rows!

∎

Theorem 6.5

$$\mathrm{OBS}_3 = \{G_{19,4}, G_{16,5}, G_{13,7}, G_{11,10}, G_{10,11}, G_{7,13}, G_{5,16}, G_{4,19}\}.$$

Proof:

We only deal with $G_{n,m}$ where $n \geq m$. We show that, for all the grids $G_{n,m}$ listed where $n \geq m$, $G_{n,m}$ is not 3-colorable but $G_{m-1,m}$ and $G_{m,m-1}$ are 3-colorable.

1. $G_{19,4}$ is not 3-colorable by Theorem 6.1. $G_{18,4}$ and $G_{19,3}$ are 3-colorable by Theorem 6.1.
2. $G_{16,5}$ is not 3-colorable by Theorem 6.1. $G_{15,5}$ and $G_{16,4}$ are 3-colorable by Theorem 6.1.
3. $G_{13,7}$ is not 3-colorable by Theorem 6.1. $G_{12,7}$ and $G_{13,6}$ are 3-colorable by Theorem 6.1.
4. $G_{11,10}$ is not 3-colorable by Theorem 6.4. $G_{10,10}$ is 3-colorable by Theorem 6.2. $G_{11,9}$ is 3-colorable by Theorem 6.1.

Table 16 indicates exactly which grids are 3-colorable. The entry for (n, m) is C if $G_{n,m}$ is 3-colorable, and N if $G_{n,m}$ is not 3-colorable. From this table one easily sees that the grids listed in this theorem are the only elements of OBS_3.

	03	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18	19	20
3	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C
4	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	N	N
5	C	C	C	C	C	C	C	C	C	C	C	C	C	N	N	N	N	N
6	C	C	C	C	C	C	C	C	C	C	C	C	C	N	N	N	N	N
7	C	C	C	C	C	C	C	C	C	C	N	N	N	N	N	N	N	N
8	C	C	C	C	C	C	C	C	C	C	N	N	N	N	N	N	N	N
9	C	C	C	C	C	C	C	C	C	C	N	N	N	N	N	N	N	N
10	C	C	C	C	C	C	C	C	N	N	N	N	N	N	N	N	N	N
11	C	C	C	C	C	C	C	N	N	N	N	N	N	N	N	N	N	N
12	C	C	C	C	C	C	C	N	N	N	N	N	N	N	N	N	N	N
13	C	C	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N
14	C	C	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N
15	C	C	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N
16	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
17	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
18	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
19	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
20	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N

Table 16: 3-Colorable Grids (C) and non 3-Colorable Grids (N)

∎

7 Which Grids Can be 4-Colored?

7.1 Results that Use our Tools

Theorem 7.1

1. *$G_{41,5}$ and $G_{5,41}$ are not 4-colorable.*
2. *$G_{31,6}$ and $G_{6,31}$ are not 4-colorable.*
3. *$G_{29,7}$ and $G_{7,29}$ are not 4-colorable.*
4. *$G_{25,9}$ and $G_{9,25}$ are not 4-colorable.*
5. *$G_{23,10}$ and $G_{10,23}$ are not 4-colorable.*
6. *$G_{22,11}$ and $G_{11,22}$ are not 4-colorable.*
7. *$G_{21,13}$ and $G_{13,21}$ are not 4-colorable.*
8. *$G_{20,17}$ and $G_{17,20}$ are not 4-colorable.*
9. *$G_{19,18}$ and $G_{18,19}$ are not 4-colorable.*
10. *$G_{41,4}$ and $G_{4,41}$ are 4-colorable (this is trivial).*
11. *$G_{40,5}$ and $G_{5,40}$ are 4-colorable.*
12. *$G_{30,6}$ and $G_{6,30}$ are 4-colorable.*
13. *$G_{28,8}$ and $G_{8,28}$ are 4-colorable.*
14. *$G_{20,16}$ and $G_{16,20}$ are 4-colorable.*

Proof:

We only consider grids $G_{n.m}$ where $n \geq m$.

1,2,3,4,5,6,7,8,9)

In Table 17 we show that several grids are not 4-colorable. For each (n, m) we use either Corollary 2.8 or 2.9. In the table we give, for each (n, m), the value of $\lceil \frac{nm}{4} \rceil$, the q, r such that $\lceil \frac{nm}{4} \rceil = qn + r$ with $0 \leq r \leq n - 1$, which corollary we use (Use), the premise of the corollary (Prem), and the arithmetic showing the premise is true (Arith).

10) $G_{41,4}$ is clearly 4-colorable.

11) $G_{40,5}$ is 4-colorable by Theorem 3.4 with $c = 4$.

12) $G_{30,6}$ is 4-colorable by Theorem 3.6 with $c = 4$ and $c' = 2$.

13) $G_{28,8}$ is 4-colorable by Theorem 3.11 with $p = 2$, $d = 3$, and $s = 1$.

14) $G_{20,16}$ is 4-colorable by Theorem 3.11 with $p = 2$, $d = 2$, and $s = 2$

∎

m	n	$\lceil \frac{nm}{4} \rceil$	q	r	Use	Prem	Arith
5	41	52	1	11	Cor 2.8	$\frac{m(m-1)}{2} < r \le n$	$10 < 11 \le 41$
6	31	47	1	16	Cor 2.8	$\frac{m(m-1)}{2} < r \le n$	$15 < 16 \le 31$
7	29	51	1	22	Cor 2.8	$\frac{m(m-1)}{2} < r \le n$	$21 < 22 \le 29$
9	25	57	2	7	Cor 2.9	$\frac{m(m-1)-2qr}{q(q-1)} < n$	$22 < 25$
10	23	58	2	12	Cor 2.9	$\frac{m(m-1)-2qr}{q(q-1)} < n$	$21 < 23$
11	22	61	2	17	Cor 2.9	$\frac{m(m-1)-2qr}{q(q-1)} < n$	$21 < 22$
13	21	69	3	6	Cor 2.9	$\frac{m(m-1)-2qr}{q(q-1)} < n$	$20 < 21$
17	20	85	4	5	Cor 2.9	$\frac{m(m-1)-2qr}{q(q-1)} < n$	$19\frac{1}{3} < 20$
18	19	86	4	10	Cor 2.9	$\frac{m(m-1)-2qr}{q(q-1)} < n$	$18\frac{5}{6} < 19$

Table 17: (m, n) such that $G_{m,n}$ is not 4-colorable

Theorem 7.2 *If $A \subseteq G_{19,17}$ and A is rectangle-free then $|A| \le 80 = \lceil \frac{19 \cdot 17}{4} \rceil - 1$. Hence $G_{19,17}$ is not 4-colorable.*

Proof: We divide the proof into cases. Every case will either conclude that $|A| \le 80$ or A cannot exist.

For $1 \le j \le 17$ let x_j be the number of elements of A in column j. We assume

$$x_1 \ge \cdots \ge x_{17}.$$

1. $6 \le x_1 \le 19$.

   By Lemma 2.16 with $x = 6$, $n = 19$, $m = 17$,

   $$|A| \le x+m-1+\text{maxrf}(n-x, m-1) \le 6+17-1+\text{maxrf}(19-6, 17-1) = 22+\text{maxrf}(13, 16).$$

   Assume, by way of contradiction, that $|A| \ge 81$. Then $\text{maxrf}(13, 16) \ge 59$ By Theorem 2.7 with $n = 16$, $m = 13$, $a = 59$, $q = 3$, $r = 11$

   $$16 \le \left\lfloor \frac{13 \times 12 - 2 \times 3 \times 11}{3 \times 2} \right\rfloor = 15.$$

   This is a contradiction.

2. There exists k, $0 \le k \le 12$, such that $x_1 = \cdots = x_k = 5$ and $x_{k+1} \le 4$. Then

   $$|A| = \sum_{j=1}^{17} x_j = (\sum_{j=1}^{k} x_j) + (\sum_{j=k+1}^{17} x_j) \le 5k + 4(17 - k) = 68 + k.$$

   Since $k \le 12$ this quantity is $\le 68 + 12 = 80$. Hence $|A| \le 80$.

3. $x_1 = x_2 = \cdots = x_{13} = 5$. Look at the grid restricted to the first 13 columns. Let B be A restricted to that grid. Note that B is a rectangle-free subset of $G_{19,13}$ of size 65. By Theorem 2.7 with $n = 19$, $m = 13$, $a = 65$, $q = 3$, and $r = 8$ we have

$$19 \leq \left\lfloor \frac{13 \times 12 - 2 \times 8 \times 3}{3 \times 2} \right\rfloor = 18.$$

This is a contradiction, hence A cannot exist.

∎

Theorem 7.3 *$G_{24,9}$ is 4-colorable.*

Proof: Table 18 shows a strong $(4, 1)$-coloring of $G_{9,6}$. Apply Theorem 3.3 with $c = 4$ and $c' = 1$ to obtain a 4-coloring of $G_{24,9}$.

1	2	2	1	2	2
1	3	3	2	1	3
1	4	4	3	3	1
2	1	4	1	4	3
3	1	2	3	1	4
4	1	3	4	2	1
4	3	1	1	3	4
2	4	1	4	1	2
3	2	1	2	4	1

Table 18: Strong 4-coloring of $4_{9,6}$.

∎

It is an open question to generalize the construction in Theorem 7.3.

7.2 Results that Needed a Computer Program

At this point in the paper the only grids whose 4-colorability is unknown are $G_{22,10}$, $G_{21,11}$, $G_{21,12}$, $G_{17,17}$, $G_{17,18}$, and $G_{18,18}$. This may seem like a computational problem that one could solve with a computer; however, the number of possible 4-coloring of (say) $G_{18,18}$ is on the order of 4^{324}. By contrast, the number of protons in the universe, also called Eddington's number, has been estimated at approximately 4^{128} [22]

The only technique we know of to show that $G_{n,m}$ is *not* c-colorable is to show that no rectangle free set of $G_{n,m}$ is of size $\geq \lceil \frac{nm}{c} \rceil$. In 2008 we obtained a rectangle free set of $G_{17,17}$ of size $74 = \lceil \frac{17 \times 17}{4} \rceil + 1$ (using a computer program). Hence we were confident that $G_{17,17}$ is 4-colorable. Using this rectangle free set as a starting point the number of possible

4-colorings would be $4^{289-74} = 4^{215}$ which is still larger than Eddington's number. For all of the other grids $G_{n,m}$ that we did not know if they were 4-colorable, a rectangle free set of size $\lceil nm/4 \rceil$ was found. Hence either they are all 4-colorable or there is a different technique to show grids are not c-colorable.

On November 30, 2009 William Gasarch (the second author on this paper) posted on his blog [6] *The* 17×17 *challenge:* if someone emails William a 4-coloring of $G_{17,17}$ then he will give them \$289.00. An earlier version of this paper was posted as well. Then the following happened:

1. Brian Hayes, a popular science writer, put the problem on his blog [11] thus exposing the problem to many more people.

2. Brad Larsen noticed that we didn't have a 4-coloring of $G_{21,11}$ and $G_{22,10}$. He then found such 4-colorings using a SAT solver which, in his words, took about 45 seconds.

3. Many people worked on finding a 4-coloring of $G_{17,17}$ (for the money! for the glory!) but could not solve it. This lead to speculation that the problem may be difficult. Evidence for this was later found [1], though by that time a 4-coloring of $G_{17,17}$ had already been found. Irony?

4. Bernd Steinbach and Christian Posthoff worked on solving the problem with SAT solvers. In a sequence of three brilliant papers they solved the problem [20, 21, 19]. This was very serious and deep research that may lead to improved SAT Solvers for other problems. They announced their result in February of 2012. See [7] for the blog post about it. Dr. Gasarch happily paid them the \$289.00.

5. Marzio De Biasi easily found an extension of the 4-coloring of $G_{17,17}$ to $G_{18,18}$ and posted it as a comment on [7]. Bernd Steinbach and Christian Posthoff had already known this coloring as well.

6. Bernd Steinbach and Christian Posthoff used their techniques to find a 4-coloring of $G_{21,12}$ and posted it as a comment on the blog post [7]. With this OBS_4 was completely known!

7. Inspired by the 17×17 challenge and the solution to it Neil Brewer and Dmitry Kamenetsky devised a contest at `http:infinitesearchspace.dyndns.org` that asked for the following: *For* $c = 1$ *to 21 find the largest* n *such that the* $n \times n$ *grid is* c*-colorable. You must also present the coloring.* This lead to a lot of interesting discussion including the following two points, one of which we use in our paper

   (a) Tom Sirgedas obtained another 4-coloring of $G_{21,12}$. To paraphrase him: *I noticed that the rectangle-free subset* A *of* $G_{21,12}$ *in (the earlier version of) the paper had the following property: If you viewed it as a* 7×3 *grid of* 3×3 *grids then in each of those* 3×3 *grids either all elements of the diagonal all in* A *or none were in* A*. I assumed that the solution would have this property. This cut down the number of* *possibilities by quite a lot. Then, I just wrote an exhaustive depth-first-search to fill the grid one color at a time, and each color one row at a time. I used a lot of pruning and bitmasks, and solutions were found in a few minutes. Unfortunately this approach seems to only work for this particular grid. It won't scale well at all.*

(b) Quimey Vivas posted a proof that if c is prime then G_{c^2,c^2} is c-colorable. Ken Berg had previously send me a proof that G_{c^2,c^2+c} is c-colorable when c is a power of a prime. That proof is in this paper as Theorem 3.14.

Theorem 7.4 *$G_{21,12}$ is 4-colorable*

Proof: Bernd Steinbach and Christian Posthoff (as a team) and Tom Sirgedas obtained a 4-coloring of $G_{21,12}$. Tom Sirgedas's coloring is in Table 19.

1	2	2	3	2	1	3	4	4	3	1	3
2	1	2	1	3	2	4	3	4	3	3	1
2	2	1	2	1	3	4	4	3	1	3	3
3	4	2	4	1	1	1	2	4	3	2	4
2	3	4	1	4	1	4	1	2	4	3	2
4	2	3	1	1	4	2	4	1	2	4	3
2	4	1	3	2	4	2	3	3	4	2	1
1	2	4	4	3	2	3	2	3	1	4	2
4	1	2	2	4	3	3	3	2	2	1	4
3	1	1	2	2	1	3	4	1	4	3	4
1	3	1	1	2	2	1	3	4	4	4	3
1	1	3	2	1	2	4	1	3	3	4	4
3	4	4	1	3	2	2	4	2	1	1	3
4	3	4	2	1	3	2	2	4	3	1	1
4	4	3	3	2	1	4	2	2	1	3	1
3	4	2	3	4	3	2	1	1	1	4	2
2	3	4	3	3	4	1	2	1	2	1	4
4	2	3	4	3	3	1	1	2	4	2	1
3	3	1	4	2	4	4	1	3	2	1	2
1	3	3	4	4	2	3	4	1	2	2	1
3	1	3	2	4	4	1	3	4	1	2	2

Table 19: A 4-coloring of $G_{21,12}$ due to Tom Sirgedas.

∎

1	2	3	3	2	2	1	1	4	4
2	1	4	2	4	4	1	3	1	3
4	2	4	3	1	1	2	3	1	4
1	1	2	2	3	3	4	4	2	1
1	4	1	1	2	3	3	2	2	3
1	4	3	4	3	1	2	3	2	2
2	1	2	1	4	1	3	4	3	2
1	3	3	4	1	4	2	2	4	3
1	4	4	3	3	2	3	2	1	2
3	3	4	4	1	2	3	4	2	1
3	2	2	1	3	4	4	2	1	3
3	4	3	2	2	1	1	4	4	2
4	3	2	4	2	3	4	3	1	1
2	2	1	4	4	1	3	3	2	4
3	2	1	3	4	3	4	1	1	2
4	4	1	2	1	4	1	2	3	3
2	1	4	3	1	2	4	1	4	3
3	4	2	1	4	2	1	3	3	1
2	4	3	1	1	3	4	2	3	4
4	3	1	2	3	2	2	4	3	1
4	3	2	3	4	1	2	1	4	1
2	1	1	3	2	4	2	4	3	4

Table 20: A 4-coloring of $2_{22,10}$ due to Brad Larsen.

Theorem 7.5 $G_{22,10}$ *is 4-colorable*

Proof: Brad Larsen obtained a 4-coloring of $G_{22,10}$: We present it in Table 20.
▮

Theorem 7.6 $G_{18,18}$ *is 4-colorable*

Proof: Bernd Steinbach and Christian Posthoff obtained a 4-coloring of $G_{18,18}$. We present the coloring in Table 21.

1	2	2	1	4	4	4	1	3	1	1	3	4	3	2	3	4	2
3	1	4	4	4	3	3	2	4	1	2	2	1	1	2	3	2	3
2	2	3	3	1	1	4	2	4	2	1	4	1	3	4	4	1	3
1	1	3	2	4	3	1	2	1	4	4	3	2	4	3	4	1	2
2	4	2	3	3	4	3	3	4	1	2	3	2	4	1	2	1	1
3	4	4	1	1	2	2	2	1	4	3	3	3	1	4	2	4	1
2	1	3	2	2	2	3	4	3	3	1	4	3	4	2	1	4	1
4	1	4	3	1	2	4	1	2	2	2	1	3	4	3	3	3	2
4	4	1	3	4	3	2	1	2	3	3	4	2	1	2	1	1	4
2	3	3	4	3	4	2	1	1	4	3	4	1	2	1	3	2	2
4	1	1	1	2	1	3	4	4	4	3	2	4	3	1	2	3	2
3	2	3	4	2	1	2	3	1	1	2	1	4	4	4	1	3	4
3	2	4	2	3	1	1	1	2	3	4	4	4	3	3	2	2	1
3	3	4	3	2	4	1	4	3	2	1	1	2	1	1	4	2	4
4	3	2	1	2	4	1	2	2	3	4	3	1	2	4	1	3	3
1	3	2	2	1	3	2	3	4	2	4	2	3	3	1	1	4	4
1	4	1	4	3	3	4	4	3	2	4	1	1	2	2	2	3	1
4	2	1	4	1	2	1	3	3	1	3	2	2	2	3	4	4	3

Table 21: A 4-coloring of $2_{18,18}$ due to Bernd Steinbach and Christian Posthoff .

▮

Theorem 7.7

$$\mathrm{OBS}_4 = \{G_{41,5}, G_{31,6}, G_{29,7}, G_{25,9}, G_{23,10}, G_{22,11}, G_{21,13}, G_{19,17}\} \bigcup$$

$$\{G_{17,19}, G_{13,21}, G_{11,22}, G_{10,23}, G_{9,25}, G_{7,29}, G_{6,31}, G_{5,41}\}$$

Proof:
We only deal with $G_{n,m}$ where $n \geq m$. We show that, for all the grids $G_{n,m}$ listed where $a \geq b$, $G_{n,m}$ is not 4-colorable but $G_{a-1,b}$ and $G_{n,m-1}$ are 4-colorable.

1. $G_{41,5}$ is not 4-colorable by Theorem 7.1. $G_{40,5}$ is 4-colorable by Theorem 7.1. $G_{41,4}$ is clearly 4-colorable.

2. $G_{31,6}$ is not 4-colorable by Theorem 7.1. $G_{30,6}$ and $G_{31,5}$ are 4-colorable by Theorem 7.1.

3. $G_{29,7}$ is not 4-colorable by Theorem 7.1. $G_{28,7}$ and $G_{29,6}$ are 4-colorable by Theorem 7.1.

4. $G_{25,9}$ is not 4-colorable by Theorem 7.1. $G_{24,9}$ is 4-colorable by Theorem 7.3. $G_{25,8}$ is 4-colorable by Theorem 7.1.

5. $G_{23,10}$ is not 4-colorable by Theorem 7.1. $G_{22,10}$ is 4-colorable by Theorem 7.5. $G_{23,9}$ is 4-colorable by Theorem 7.3.

6. $G_{22,11}$ is not 4-colorable by Theorem 7.1. $G_{22,10}$ is 4-colorable by Theorem 7.5. $G_{21,11}$ is 4-colorable by Theorem 7.4.

7. $G_{21,13}$ is not 4-colorable by Theorem 7.1. $G_{20,13}$ is 4-colorable by Theorem 7.1. $G_{21,12}$ is 4-colorable by Theorem 7.4.

8. $G_{19,17}$ is not 4-colorable by Theorem 7.2. $G_{18,17}$ is 4-colorable by Theorem 7.6. $G_{19,16}$ is 4-colorable by Theorem 7.1.

The following chart indicates exactly which grids are 4-colorable. The entry for (n,m) is C if $G_{n,m}$ is 4-colorable, and N if $G_{n,m}$ is not 4-colorable. From the chart one easily sees that the grids listed in this theorem are the only elements of OBS_4.

∎

8 Application to Bipartite Ramsey Numbers

We state the Bipartite Ramsey Theorem. See [9] for history, details, and proof.

Def 8.1 A complete bipartite graph, $G = (V_1, V_2, E)$, is a bipartite graph such that for any two vertices, $v_1 \in V_1$ and $v_2 \in V_2$, (v_1, v_2) is an edge in G. The complete bipartite graph with partitions of size $|V_1| = a$ and $|V_2| = b$, is denoted $K_{a,b}$.

Theorem 8.2 *For all a, c there exists $n = BR(a,c)$ such that for all c-colorings of the edges of $K_{n,n}$ there will be a monochromatic $K_{a,a}$.*

The following theorem is easily seen to be equivalent to this.

Theorem 8.3 *For all a, c there exists $n = BR(a,c)$ so that for all c-colorings of $G_{n,n}$ there will be a monochromatic $a \times a$ submatrix.*

	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18	19	20	21
8	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C
9	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C
10	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C
11	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C
12	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C
13	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	N
14	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	N
15	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	N
16	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	N
17	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	N	N	N
18	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	N	N	N
19	C	C	C	C	C	C	C	C	C	C	C	C	C	N	N	N	N	N
20	C	C	C	C	C	C	C	C	C	C	C	C	C	N	N	N	N	N
21	C	C	C	C	C	C	C	C	C	N	N	N	N	N	N	N	N	N
22	C	C	C	C	C	C	C	N	N	N	N	N	N	N	N	N	N	N
23	C	C	C	C	C	C	N	N	N	N	N	N	N	N	N	N	N	N
24	C	C	C	C	C	C	N	N	N	N	N	N	N	N	N	N	N	N
25	C	C	C	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N
26	C	C	C	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N
27	C	C	C	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N
28	C	C	C	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N
29	C	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
30	C	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
31	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
32	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
33	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
34	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
35	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
36	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
37	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
38	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
39	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
40	C	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
41	C	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N

Table 22: 4-Colorable Grids (C) and non 4-Colorable Grids (N)

In this paper we are c-coloring $G_{n,m}$ and looking for a 2×2 monochromatic submatrix. We have the following theorems which, except where noted, seem to be new.

Theorem 8.4

1. *$BR(2,2) = 5$. (This was also shown in [14].)*
2. *$BR(2,3) = 11$.*
3. *$BR(2,4) = 19$.*
4. *$BR(2,c) \le c^2 + c$.*
5. *If p is a prime and $s \in \mathbb{N}$, then $BR(2,p^s) > p^{2s}$.*
6. *For almost all c, $BR(2,c) \ge c^2 - 2c^{1.525} + c^{1.05}$.*

Proof:
1) By Theorem 5.1, $G_{5,5}$ is not 2-colorable and $G_{4,4}$ is 2-colorable.
2) By Theorem 6.4, $G_{11,11}$ is not 3-colorable. By Theorem 6.2 $G_{10,10}$ is 3-colorable.
3) By Theorem 7.2, $G_{19,19}$ is not 4-colorable. By Theorem 7.6 $G_{18,18}$ is 4-colorable.
4) By Theorem 4.1, G_{c^2+c,c^2+c} is not c-colorable.
5) By Theorem 3.11, $G_{cr,cm}$ is c-colorable where $c = p^s$, $r = p^s$, and $m = \frac{p^{2s}-1}{p^s-1}$. Note that $m \ge p^s$. Hence $G_{p^{2s},p^{2s}}$ is p^s-colorable.
6) Baker, Harman, and Pintz [3] (see [12] for a survey) showed that for almost all c, there is a prime between c and $c - c^{0.525}$. Let p be that prime. By part 5 with $s = 1$, $BR(2,p) \ge p^2$. Hence

$$BR(2,c) \ge BR(2,p) \ge p^2 \ge (c - c^{0.525})^2 = c^2 - 2c^{1.525} + c^{1.05}.$$

$\blacksquare$

9 Open Questions

1. Refine our tools so that our ugly proofs can be corollaries of our tools.
2. Find an algorithm that will, given c, find OBS_c or $|\mathrm{OBS}_c|$ quickly.
3. We know that $2\sqrt{c}(1 - o(1)) \le |\mathrm{OBS}_c| \le 2c^2$. Bring these bounds closer together.
4. All of our results of the form $G_{n,m}$ *is not c-colorable* have the same type of proof: show that there is no rectangle free subset of $G_{n,m}$ of size $\lceil ab/c \rceil$. Either
    - show that if a grid $G_{n,m}$ has a rectangle free set of size $\lceil nm/c \rceil$ then it is c-colorable, or

- develop some other technique to show grids are not c-colorable.

5. Find OBS_5 and beyond!

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11 Appendix: Exact Values of $\mathrm{maxrf}(n,m)$ for $0 \le n \le 6$, $m \le n$

Lemma 11.1

0) For $m \ge 0$, $\mathrm{maxrf}(0,m) = 0$.

1) For $m \ge 1$, $\mathrm{maxrf}(1,m) = m$.

2) For $m \ge 2$, $\mathrm{maxrf}(2,m) = m+1$.

3) For $m \ge 3$, $\mathrm{maxrf}(3,m) = m+3$.

4)

$$\mathrm{maxrf}(4,m) = \begin{cases} m+5 & \textit{if } 4 \le m \le 5 \\ m+6 & \textit{if } m \ge 6 \end{cases}$$

5)

$$\text{maxrf}(5, m) = \begin{cases} 12 \ \textit{if} \ m = 5 \\ m + 8 \ \textit{if} \ 6 \leq m \leq 7 \\ m + 9 \ \textit{if} \ 8 \leq m \leq 9 \\ m + 10 \ \textit{if} \ m \geq 10 \end{cases}$$

6)

$$\text{maxrf}(6, m) = \begin{cases} 2m + 4 \ \textit{if} \ 6 \leq m \leq 7 \\ 19 \ \textit{if} \ m = 8 \\ m + 12 \ \textit{if} \ 9 \leq m \leq 10 \\ m + 13 \ \textit{if} \ 11 \leq m \leq 12 \\ m + 14 \ \textit{if} \ 13 \leq m \leq 14 \\ m + 15 \ \textit{if} \ m \geq 15 \end{cases}$$

Proof:

Theorem 2.7 will provide all of the upper bounds. The lower bounds are obtained by actually exhibiting rectangle-free sets of the appropriate size. We do this for the case of $\text{maxrf}(6, m)$. Our technique applies to all of the other cases.

Case 1: $\text{maxrf}(6, m)$ **where** $6 \leq m \leq 7$ **and** $m = 8$**:** Fill the first four columns with 3 elements (all pairs overlapping). Each column of 3 blocks exactly $\binom{3}{2} = 3$ of the possible $\binom{6}{3} = 15$ ordered pairs, hence 12 are blocked. Hence we can fill the next $15 - 12 = 3$ columns with two elements each, and the remaining column (if $m = 8$) with 1 element. The picture below shows the result for $\text{maxrf}(6, 8) = 19$; however, if you just look at the first 6 (7) columns you get the result for $\text{maxrf}(6, 6)$ ($\text{maxrf}(6, 7)$).

R		R		R			R
R			R		R		
R	R					R	
	R	R			R		
	R		R	R			
		R	R			R	

Case 2: $\text{maxrf}(6, m)$ **where** $9 \leq m \leq 10$**:** Fill the first three columns with 3 elements each (all pairs overlapping). Each column of 3 blocks exactly $\binom{3}{2} = 3$ of the possible $\binom{6}{3} = 15$ ordered pairs, hence 9 are blocked. Hence we can fill the next $15 - 9 = 6$ columns with two elements each and the remaining column (if $m = 10$) with 1 element. The picture below shows the result for $\text{maxrf}(6, 10) = 22$; however, if you just look at the first 9 columns you get the result $\text{maxrf}(6, 9) = 21$.

R		R		R					
R					R		R	R	
R	R					R			
	R	R			R				
	R		R	R			R		
		R	R			R		R	R

Case 3: $\mathrm{maxrf}(6, m)$ **where** $11 \le m \le 12$**:** Fill the first two columns with 3 elements each (they overlap). Each column of 3 blocks exactly $\binom{3}{2} = 3$ of the possible $\binom{6}{3} = 15$ ordered pairs, hence 6 are blocked. Hence we can fill the next $15 - 6 = 9$ columns with two elements each and the remaining column (if $m = 12$) with 1 element. The picture below shows the result for $\mathrm{maxrf}(6, 12) = 25$; however, if you just look at the first 11 columns you get the result $\mathrm{maxrf}(6, 11) = 24$.

R		R		R						R	
R					R		R	R			
R	R					R					
	R	R			R				R		
	R		R	R			R				
			R			R		R	R	R	R

Case 4: $\mathrm{maxrf}(6, m)$ **where** $13 \le m \le 14$**:** Fill the first column with 3 elements. This column of 3 blocks exactly $\binom{3}{2} = 3$ of the possible $\binom{6}{3} = 15$ ordered pairs. Hence we can fill the next $15 - 3 = 12$ columns with two elements each and the remaining column (if $m = 14$) with 1 element. We omit the picture.

Case 5: $\mathrm{maxrf}(6, m)$ **where** $m \ge 15$**:** Fill the first $\binom{6}{2} = 15$ columns with two elements each in a way so that each column has a distinct pair. Fill the remaining $m - 15$ columns with one element each. The result is a rectangle-free set of size $30 + m - 15 = m + 15$. ▮

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