Homework 1, Morally Due Tue Feb 5, 2013
COURSE WEBSITE: http://www.cs.umd.edu/~gasarch/858/S13.html
(The symbol before gasarch is a tilde.)

1. (10 points) What is your name? Write it clearly. Staple your HW. When is the midterm (give Date and Time)? If you cannot make it in that day/time see me ASAP. Join the Piazza group for the course. The codename is cmsc858. Look at the link on the class webpage about projects. Come see me about a project. READ the note on the class webpage that say THIS YOU SHOULD READ that you haven't already read.

2. (20 points) Recall that the a-ary infinite Ramsey Theorem dealt with colorings of $\binom{\mathsf{N}}{a}$. We have only dealt with $a \ge 2$.

   (a) Formulate the 1-ary infinite Ramsey Theorem, for c colors, and prove it.

   (b) Formulate the ω-ary infinite Ramsey Theorem. (Extra Credit- prove or disprove it.)

   SOLUTION TO PROBLEM 2

   The key to this problem was to DEFINE homog sets.

   1) Given $COL : \binom{\mathsf{N}}{1} \to [2]$, a homog set is a set of numbers that are all colored the same. Hence the statement is:

   For all $COL : \binom{\mathsf{N}}{1} \to [2]$ *there is an infinite subset* $A \subseteq \mathsf{N}$ *such that all the elements of* A *are colored the same.*

   OR, if you defined homog you could just say

   For all $COL : \binom{\mathsf{N}}{1} \to [2]$ *there is an infinite homog subset* $A \subseteq \mathsf{N}$.

   2) Given $COL : \binom{\mathsf{N}}{\omega} \to [2]$, a homog set is an infinite set A such that all infinite subsets of A are colored the same. Hence the statement is:

   For all $COL : \binom{\mathsf{N}}{\omega} \to [2]$ *there is an infinite subset* $A \subseteq \mathsf{N}$ *such that all subsets of* A *are colored the same.*

   OR, if you defined homog you could just say

   For all $COL : \binom{\mathsf{N}}{\omega} \to [2]$ *there is an infinite homog subset* $A \subseteq \mathsf{N}$.

3. (40 points) State and prove (rigorously) the c-color a-ary Ramsey Theorem. Your statement should start out *for all* $a \geq 1$, *for all* $c \geq 1, \ldots$. The proof should be by induction on a with the base case being $a = 1$.

   Omitted- very similar to what we did in class.

4. (40 points) Show (rigorously) that there exists a computable 2-coloring of $\binom{\mathsf{N}}{2}$ with no c.e.-in-$HALT$ homog set. (HINT- the proof is very similar to the one you saw in class. Instead of looking at $W_{e,s}$ you look at $W_{e,s}^{HALT_s}$.) (NOTE- I ALLOW THE FOLLOWING TECHNICAL ASSUMPTION: if W_e^{HALT} is a c.e.-in-HALT set then it can only change its mind finitely often on any one number. Formally: For every x there is an $s_0 \in \mathsf{N}$ such that one of the two holds:

   (1) $(\forall s \geq s_0)[x \in W_{e,s}^{HALT_s}]$

   (2) $(\forall s \geq s_0)[x \notin W_{e,s}^{HALT_s}]$.

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   The construction is similar to the one I did in class: just replace $W_{e,s}$ with $W_{e,s}^{K_s}$. But the proof that it works needs some serious changes.

   I do the proof as though its the proof I did in class and then say where it differs.

   We show that each requirement is eventually satisfied.

   For pedagogue we first look at R_1.

   If W_1^K is finite then R_1 is satisfied.

   Assume W_1^K is infinite. We show that R_1^K is satisfied. Let $x < y$ be the least two elements in W_1^K. Let s_0 be the least number such that $x, y \in W_{1,s_0}^{K_{s_0}}$.

   NO NO NO!!!!- It could be that for some later $s \geq s_0$ we have $x, y \notin W_{1,s}^{K_s}$. ALSO it is possible that for some later $s \geq s_0$ some SMALLER values x', y' are in $W_{1,s}^{K_s}$ and they will be the ones whose edges to s get colored.

   It is ESSENTIAL to take x_0 such that

   - $x, y \in W_{1,s_0}^{K_{s_0}}$
   - $(\forall s \geq s_0)[x, y \in W_{1,s}^{K_s}]$.

- $(\forall s \ge s_0)[0, \ldots, x-1, x+1, x+2, \ldots, y-1 \notin W_{1,s}^{K_s}]$.

NOW we have that, for ALL $s \ge s_0$:

$COL(x,s) = RED$

$COL(y,s) = BLUE$

Since W_1^K is infinite there is SOME $s \ge s_0$ with $s \in W_{e,s}^{K_s}$. Hence $x, y, s \in W_1^K$ and show that W_1^K is NOT homogenous.

Can we show R_2 is satisfied the same way? Yes but with a caveat- we won't use the least two elements of W_2^K. We'll use the least two elements of W_2^K that are bigger than the least two elements of W_1^K. We now do this rigorously and more generally.

Claim: For all e, R_e is satisfied:

Proof: Fix e. If W_e^K is finite then R_e is satisfied.

Assume W_e^K is infinite. We show that R_e is satisfied. Let $x_1 < x_2 < \cdots < x_{2e}$ be the first (numerically) $2e$ elements of W_e^K. Let s_0 be the least number such that

- $x_1, \ldots, x_e \in W_{1,s_0}^{K_{s_0}}$
- $(\forall s \ge s_0)[x_1, \ldots, x_e \in W_{1,s}^{K_s}]$.
- $(\forall s \ge s_0)(\forall z \in [x_{2e}] - \{x_1, \ldots, x_{2e}\}[z \notin W_{1,s}^{K_s}]$.

KEY: for all $s \ge s_0$, during stage s, the requirements $R_1, \ldots, R_{e-1}$ may define $COL(x,s)$ for some of the $x \in \{x_1, \ldots, x_{2e}\}$. But they will NOT define $COL(x,s)$ for ALL of those x. Why? Because R_i only defines $COL(x,s)$ for at most TWO of those x's, and there are $e-1$ such i, so at most $2e-2$ of those x's have $COL(x,s)$ defined. Hence there will exist x, y such that R_e gets to define $COL(x,s)$ and $COL(y,s)$. Furthermore, they will always be the SAME x, y since the R_i with $i < e$ have already made up their minds about the x in $\{x_1, \ldots, x_{2e}\}$.

UPSHOT: There exists $x, y \in W_e^K$ such that, for all $s \ge s_0$,

$COL(x,s) = RED$

$COL(y,s) = BLUE$

Since W_e^K is infinite there is SOME $s \ge s_0$ with $s \in W_e^K$. Hence $x, y, s \in W_e^K$ and show that W_e is NOT homogenous.