Three Proofs of the Hypergraph Ramsey Theorem (An Exposition)

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Abstract

Ramsey, Erdős-Rado, and Conlon-Fox-Sudakov have given proofs of the 3-hypergraph Ramsey Theorem with better and better upper bounds on the 3-hypergraph Ramsey numbers. Ramsey and Erdős-Rado also prove the a-hypergraph Ramsey Theorem. Conlon-Fox-Sudakov note that their upper bounds on the 3-hypergraph Ramsey Numbers, together with a recurrence of Erdős-Rado (which was the key to the Erdős-Rado proof), yield improved bounds on the a-hypergraph Ramsey numbers. We present all of these proofs and state explicit bounds for the 2-color case and the c-color case. We give a more detailed analysis of the construction of Conlon-Fox-Sudakov and hence obtain a slightly better bound.

1 Introduction

The 3-hypergraph Ramsey numbers $R(3,k)$ were first shown to exist by Ramsey [8]. His upper bounds on them were enormous. Erdős-Rado [3] obtained much better bounds, namely $R(3,k) \leq$

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$2^{2^{4k}}$. Recently Conlon-Fox-Sudakov [2] have obtained $R(3,k) \le 2^{2^{(2+o(1))k}}$. We present all three proofs. For the Conlon-Fox-Sudakov proof we give a more detailed analysis that required a non-trivial lemma, and hence we obtain slightly better bounds. Before starting the second and third proofs we will discuss why they improve the prior ones.

We also present extensions of all three proofs to the a-hypergraph case. The first two are known proofs and bounds. The Erdős-Rado proof gives a recurrence to obtain a-hypergraph Ramsey Numbers from $(a-1)$-hypergraph Ramsey Numbers. As Conlon-Fox-Sudakov note, this recurrence together with their improved bound on $R(3,k)$, yield better upper bounds on the a-hypergraph Ramsey Numbers. Can the Conlon-Fox-Sudakov method itself be extended to a proof of the a-hypergraph Ramsey Theorem? It can; however (alas), this does not seem to lead to better upper bounds. We include this proof in the appendix in the hope that someone may improve either the construction or the analysis to obtain better bounds on the a-hypergraph Ramsey Numbers.

For all of the proofs, the extension to c colors is routine. We present the results as notes; however, we leave the proofs as easy exercises for the reader.

2 Notation and Ramsey's Theorem

Def 2.1 Let X be a set and $a \in \mathsf{N}$. Then $\binom{X}{a}$ is the set of all subsets of X of size a.

Def 2.2 Let $a, n \in \mathsf{N}$. The *complete a-hypergraph on n vertices*, denoted K_n^a, is the hypergraph with vertex set $V = [n]$ and edge set $E = \binom{[n]}{a}$

Notation 2.3 In this paper a *coloring of a graph or hypergraph* always means a coloring of the *edges*. We will abbreviate $COL(\{x_1, \ldots, x_a\})$ by $COL(x_1, \ldots, x_a)$. We will refer to a c-coloring of the edges of the complete hypergraph K_n^a as a c-coloring of $\binom{[n]}{a}$.

Def 2.4 Let $a \geq 1$. Let COL be a c-coloring of $\binom{[n]}{a}$. A set of vertices H is *a-homogeneous for COL* if every edge in $\binom{H}{a}$ is the same color. We will drop the *for COL* when it is understood. We will drop the a when it is understood.

Convention 2.5 When talking about 2-colorings will often denote the colors by RED and BLUE.

Note 2.6 In Definition 2.4 we allow $a = 1$. Note that a c-coloring of $\binom{[n]}{1}$ is just a coloring of the numbers in $[n]$. A homogenous subset H is a subset of points that are all colored the same. Note that in this case the edges are 1-subsets of the points and hence are identified with the points.

Def 2.7 Let $a, c, k \in \mathsf{N}$. Let $R(a, k, c)$ be the least n such that, for all c-colorings of $\binom{[n]}{a}$ there exists an a-homogeneous set $H \in \binom{[n]}{k}$. We denote $R(a, k, 2)$ by $R(a, k)$. We have not shown that $R(a, k, c)$ exists; however, we will.

We state Ramsey's theorem for 1-hypergraphs (which is trivial) and for 2-hypergraphs (just graphs). The 2-hypergraph case (and the a-hypergraph case) is due to Ramsey [8] (see also [4, 6, 7]). The bound we give on $R(2, k)$ seems to be folklore (see [6]).

Def 2.8 The expression $\omega(1)$ means a function that goes to infinity monotonically. For example, $\lfloor \lg \lg n \rfloor$.

The following are well known.

Theorem 2.9 *Let $k \in \mathsf{N}$ and $c \geq 2$.*

1. $R(1, k) = 2k - 1$.
2. $R(1, k, c) = ck - c + 1$.
3. $R(2, k) \leq \binom{2k-2}{k-1} \leq 2^{2k-0.5\lg(k-1)-\Omega(1)}$.

4. $R(2,k,c) \le \frac{(c(k-1))!}{(k-1)!)^c} \le c^{ck-0.5\log_c(k-1)+O(c)}$.

5. *For all n, for every 2-coloring of $\binom{[n]}{2}$, there exists a 2-homogenous set H of size at least $\frac{1}{2}\lg n + \omega(1)$. (This follows from Part 3 easily. In fact, all you need is $R(2,k) \le 2^{2k-\Omega(1)}$.)*

Note 2.10 Theorem 2.9.2 has an elementary proof. A more sophisticated proof, by David Conlon [1] yields $R(2,k) \le k^{-E\frac{\log k}{\log\log k}}\binom{2k}{k}$, where E is some constant. A simple probabilistic argument shows that $R(2,k) \ge (1+o(1))\frac{1}{e\sqrt{2}}k2^{k/2}$. A more sophisticated argument by Spencer [9] (see [6]), that uses the Lovasz Local Lemma, shows $R(2,k) \ge (1+o(1))\frac{\sqrt{2}}{e}k2^{k/2}$.

We state Ramsey's theorem on a-hypergraphs [8] (see also [6, 7]).

Theorem 2.11 *Let $a,k,c \in \mathsf{N}$. For all $k \in \mathsf{N}$, $R(a,k,c)$ exists.*

3 Summary of Results

We will need both the tower function and Knuth's arrow notation to state the results.

Notation 3.1

$$c \uparrow^a k = \begin{cases} ck & \text{if } a = 0, \\ c^k, & \text{if } a = 1, \\ 1, & \text{if } k = 0, \\ c \uparrow^{a-1} (c \uparrow^a (k-1)), & \text{otherwise.} \end{cases}$$

Def 3.2 We define TOW which takes on a variable number of arguments.

1. $\mathrm{TOW}_c(b) = c^b$.

2. $\mathrm{TOW}_c(b_1,\ldots,b_L) = c^{b_1 \mathrm{TOW}_c(b_2,\ldots,b_L)}$.

When c is not stated it is assumed to be 2.

Example 3.3

1. $\mathrm{TOW}(2k) = 2^{2k}$.

2. $\mathrm{TOW}(1, 4k) = 2^{2^{4k}}$.

3. $\mathrm{TOW}(1) = 2$, $\mathrm{TOW}(1, 1) = 2^2$, $\mathrm{TOW}(1, 1, 1) = 2^{2^2}$.

The list below contains both who proved what bounds and the results we will prove in this paper.

1. Ramsey's proof [8] yields:

   (a) $R(3, k) \le 2 \uparrow^2 (2k - 1) = \mathrm{TOW}(1, \ldots, 1)$ where the number of 1's is $2k - 1$.

   (b) $R(a, k) \le 2 \uparrow^{a-1} (2k - 1)$.

2. The Erdős-Rado [3] proof yields:

   (a) $R(3, k) \le 2^{2^{4k-\lg(k-2)}}$.

   (b) $R(a, k) \le 2^{\binom{R(a-1,k-1)+1}{a-1}} + a - 2$.

   (c) Using the recurrence they obtain the following: For all $a \ge 4$, $R(a, k) \le \mathrm{TOW}(1, a - 1, a - 2, \ldots, 3, 4k - \lg(k - a + 1) - 4(a - 3)))$.

3. The Conlon-Fox-Sudakov [2] proof yields:

   (a) $R(3, k) \le 2^{B(k-1)^{1/2} 2^{2k}}$ where $B = \left(\frac{e}{\sqrt{2\pi}}\right)^3 \sim 1.28$.

   (b) If you combine this with the recurrence obtained by Erdős-Rado then one obtains:

      i. $R(3, k) \le \mathrm{TOW}(B(k - 1)^{1/2}, 2^{2k})$.

      ii. $R(4, k) \le \mathrm{TOW}(1, 3B(k - 2)^{1/2}, 2^{2k-2})$.

      iii. $R(5, k) \le \mathrm{TOW}(1, 4, 3B(k - 3)^{1/2}, 2^{2k-4})$.

iv. For all $a \geq 6$, for almost all k,

$$R(a,k) \leq \mathrm{TOW}(1, a-1, a-2, \ldots, 4, 3B(k-a+2)^{1/2}, 2^{2k-2a+6}).$$

4. The Appendix contains an alternative proof of the a-hypergraph Ramsey Theorem based on the ideas of Conlon-Fox-Sudakov. Since it does not yield better bounds we do not state the bounds here.

Notation 3.4 PHP stands for Pigeon Hole Principle.

We will need the following lemma whose easy proof we leave to the reader.

Lemma 3.5 *For all $b, b_1, \ldots, b_L \in \mathsf{N}$ the following hold.*

1. $\mathrm{TOW}(b_1, \ldots, b_i, b_{i+1}, b_{i+2} \ldots, b_L) \leq \mathrm{TOW}(b_1, \ldots, 1, b_{i+1} + \lg(b_i), b_{i+2}, \ldots, b_L)$.
2. $\mathrm{TOW}(b_1, \ldots, b_L)^b = \mathrm{TOW}(bb_1, b_2, \ldots, b_L)$.
3. $(1+\delta)\mathrm{TOW}(b_1, \ldots, b_L) \leq \mathrm{TOW}(b_1, b_2, \ldots, b_L + \delta)$.
4. $(1+\delta)\mathrm{TOW}(b_1, \ldots, b_L)^b \leq \mathrm{TOW}(bb_1, b_2, \ldots, b_L + \delta)$. *(This follows from 1 and 2.)*
5. $2^{\mathrm{TOW}(b_1, \ldots, b_L)} = \mathrm{TOW}(1, b_1, \ldots, b_L)$.
6. $2^{(1+\delta)\mathrm{TOW}(b_1, \ldots, b_L)^b} \leq \mathrm{TOW}(1, bb_1, b_2, \ldots, b_L + \delta)$. *(This follows from 4 and 5.)*
7. $\lg^{(c)}(\mathrm{TOW}(1, \ldots, 1)) = 1$ *(there are c 1's).*

4 Ramsey's Proof

Theorem 4.1 *For almost k $R(3,k) \leq 2 \uparrow^2 (2k-1) = \mathrm{TOW}(1, \ldots, 1)$ where there are $2k-1$ 1's.*

Proof:

Let n be a number to be determined. Let COL be a 2-coloring of $\binom{[n]}{3}$. We define a sequence of vertices,

$$x_1, x_2, \ldots, x_{2k-1}.$$

Here is the basic idea: Let $x_1 = 1$. This induces the following coloring of $\binom{[n]-\{1\}}{2}$:

$$COL^*(x,y) = COL(x_1, x, y).$$

By Theorem 2.9 there exists a 2-homogeneous set for COL^* of size $\frac{1}{2}\lg n + \omega(1)$. Keep that 2-homogeneous set and ignore the remaining points. Let x_2 be the least vertex that has been kept (bigger than x_1). Repeat the process.

We describe the construction formally.

CONSTRUCTION

$V_0 = [n]$

Assume $1 \le i \le 2k-1$ and that V_{i-1}, $x_1, x_2, \ldots, x_{i-1}$, $c_1, \ldots, c_{i-1}$ are all defined. We define x_i, COL^*, V_i, and c_i:

$$\begin{aligned}
x_i = &\quad \text{the least number in } V_{i-1} \\
V_i = &\quad V_{i-1} - \{x_i\} \text{ (We will change this set without changing its name.)} \\
COL^*(x,y) = &\quad COL(x_i, x, y) \text{ for all } \{x,y\} \in \binom{V_i}{2} \\
V_i = &\quad \text{the largest 2-homogeneous set for } COL^* \\
c_i = &\quad \text{the color of } V_i
\end{aligned}$$

KEY: for all $y, z \in V_i$, $COL(x_i, y, z) = c_i$.

END OF CONSTRUCTION

When we derive upper bounds on n we will show that the construction can be carried out for $2k-1$ stages. For now assume the construction ends.

We have vertices

$$x_1, x_2, \ldots, x_{2k-1}$$

and associated colors

$$c_1, c_2, \ldots, c_{2k-1}.$$

There are only two colors, hence, by PHP, there exists $i_1, \ldots, i_k$ such that $i_1 < \cdots < i_k$ and

$$c_{i_1} = c_{i_2} = \cdots = c_{i_k}$$

We take this color to be RED. We show that

$$H = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}.$$

is 3-homogenous for COL. For notational convenience we show that $COL(x_{i_1}, x_{i_2}, x_{i_3}) =$ RED. The proof for any 3-set of H is similar. By the definition of c_{i_1} $(\forall A \in \binom{V_{i_1} - \{x_{i_1}\}}{2})[COL(A \cup \{x_{i_1})\}) = c_i]$ In particular

$$COL(x_{i_1}, x_{i_2}, x_{i_3}) = c_{i_1} = \text{RED}.$$

We now see how large n must be so that the construction can be carried out. By Theorem 2.9, if k is large, at every iteration V_i gets reduced by a logarithm, cut in half, and then an $\omega(1)$ is added. Using this it is easy to show that, for almost all k,

$$|V_j| \geq \frac{1}{2}(\lg^{(j)} n) + \omega(1).$$

We want to run this iteration $2k-1$ times Hence we need

$$|V_{2k-1}| \geq \frac{1}{2}\log_2^{(2k-1)} n + \omega(1) \geq 1.$$

We can take $n = \mathrm{TOW}(1, \ldots, 1)$ where 1 appears $2k-1$ times, and use Lemma 3.5. ∎

Note 4.2 The proof of Theorem 4.1 generalizes to c-colors to yield

$$R(3, k, c) \leq c \uparrow^2 (ck - c + 1) = \mathrm{TOW}_c(1, \ldots, 1)$$

where the number of 1's is $ck - c + 1$.

We now prove Ramsey's Theorem for a-hypergraphs.

Theorem 4.3 *For all $a \geq 1$, for all $k \geq 1$, $R(a, k) \leq 2 \uparrow^{a-1} (2k-1)$.*

Proof:

We prove this by induction on a. Note that when we have the theorem for a we have it for a and for *all* $k \geq 1$.

Base Case: If $a = 1$ then, for all $k \geq 1$, $R(1, k) = 2k - 1 \leq 2 \uparrow^0 (2k-1) = 4k - 2$.

Induction Step: We assume that, for all k, $R(a-1, k) \leq 2 \uparrow^{a-2} (2k-1)$.

Let $k \geq 1$. Let n be a number to be determined later. Let COL be a 2-coloring of $\binom{[n]}{a}$. We show that there is an a-homogenous set for COL of size k.

CONSTRUCTION

$V_0 =]n]$.

Assume $1 \leq i \leq 2k-1$ and that $V_{i-1}, x_1, x_2, \ldots, x_{i-1}, c_1, \ldots, c_{i-1}$ are all defined. We define x_i, COL^*, V_i, and c_i:

$$
\begin{aligned}
x_i = &\quad \text{the least number in } V_{i-1} \\
V_i = &\quad V_{i-1} - \{x_i\} \text{ (We will change this set without changing its name.)} \\
COL^*(A) = &\quad COL(x_i \cup A) \text{ for all } A \in \binom{V_i}{a-1}) \\
V_i = &\quad \text{the largest } a-1\text{-homogeneous set for } COL^* \\
c_i = &\quad \text{the color of } V_i
\end{aligned}
$$

KEY: For all $1 \le i \le 2k-1$, $(\forall A \in \binom{V_i}{a-1})[COL(A \cup x_i) = c_i]$

END OF CONSTRUCTION

When we derive upper bounds on n we will show that the construction can be carried out for $2k-1$ stages. For now assume the construction ends.

We have vertices

$$x_1, x_2, \dots, x_{2k-1}$$

and associated colors

$$c_1, c_2, \dots, c_{2k-1}.$$

There are only two colors, hence, by PHP, there exists $i_1, \dots, i_k$ such that $i_1 < \cdots < i_k$ and

$$c_{i_1} = c_{i_2} = \cdots = c_{i_k}$$

We take this color to be RED. We show that

$$H = \{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}.$$

is a-homogenous for COL. For notational convenience we show that $COL(x_{i_1}, \ldots, x_{i_a}) = \text{RED}$. The proof for any a-set of H is similar. By the definition of c_{i_1} $(\forall A \in \binom{V_{i_1}}{a-1})[COL(A \cup x_{i_1}) = c_i]$ In particular

$$COL(x_{i_1}, \ldots, x_{i_a}) = c_{i_1} = \text{RED}.$$

We show that if $n = 2 \uparrow^{a-1} (2k-1)$ then the construction can be carried out for $2k-1$ stages.

Claim 1: For all $0 \le i \le 2k-1$, $|V_i| \ge 2 \uparrow^{a-1} (2k-(i+1))$.

Proof of Claim 1: We prove this claim by induction on i. For the base case note that

$$|V_0| = n = 2 \uparrow^{a-1} (2k-1).$$

Assume $|V_{i-1}| \ge 2 \uparrow^{a-1} (2k-i)$. By the definition of the uparrow function and by the inductive hypothesis of the theorem,

$$|V_{i-1}| \ge 2 \uparrow^{a-1} (2k-i) = 2 \uparrow^{a-2} (2 \uparrow^{a-1} (2k-(i+1))) \ge R(a-1, 2 \uparrow^{a-1} (2k-(i+1))).$$

By the construction V_i is the result of applying the $(a-1)$-ary Ramsey Theorem to a 2-coloring of $\binom{V_{i-1}}{a}$. Hence $|V_i| \ge 2 \uparrow^{a-1} (2k-(i+1))$.

End of Proof of Claim 1

By Claim 1 if $n = 2 \uparrow^{a-1} (2k-1)$ then the construction can be carried out for $2k-1$ stages. Hence $R(a,k) \le 2 \uparrow^{a-1} (2k-1)$. ∎

The proof of Theorem 4.1 is actually an ω^2-induction that is similar in structure to the original proof of van der Warden's theorem [5, 6, 10].

Note 4.4 The proof of Theorem 4.3 generalizes to c colors yielding

$$R(a,k,c) \le c \uparrow^{a-1} (ck-c+1).$$

5 The Erdős-Rado Proof

Why does Ramsey's proof yield such large upper bounds? Recall that in Ramsey's proof we do the following:

- Color a *node* by using Ramsey's theorem (on graphs). This cuts the number of nodes down by a log (from m to $\Theta(\log m)$). This is done $2k-1$ times.
- After the nodes are colored we use PHP once. This will cut the number of nodes in half.

The key to the large bounds is the number of times we use Ramsey's theorem. The key insight of the proof by Erdős and Rado [3] is that they use PHP many times but Ramsey's theorem only once. In summary they do the following:

- Color an *edge* by using PHP. This cuts the number of nodes in half. This is done $R(2,k-1)+1$ times.
- After *all* the edges of a complete graph are colored we use Ramsey's theorem. This will cut the number of nodes down by a log.

We now proceed formally.

Theorem 5.1 *For almost all k, $R(3,k) \le 2^{2^{4k-\lg(k-2)}}$.*

Proof:

Let n be a number to be determined. Let COL be a 2-coloring of $\binom{[n]}{3}$. We define a sequence of vertices,

$$x_1, x_2, \ldots, x_{R(2,k-1)+1}.$$

Recall the definition of a 1-homogeneous set for a coloring of singletons from the note following Definition 2.4. We will use it here.

Here is the intuition: Let $x_1 = 1$. Let $x_2 = 2$. The vertices x_1, x_2 induces the following coloring of $\{3, \ldots, n\}$.

$$COL^*(y) = COL(x_1, x_2, y).$$

Let V_1 be a 1-homogeneous for COL^* of size at least $\frac{n-2}{2}$. Let $COL^{**}(x_1, x_2)$ be the color of V_1. Let x_3 be the least vertex left (bigger than x_2).

The number x_3 induces *two* colorings of $V_1 - \{x_3\}$:

$$(\forall y \in V_1 - \{x_3\})[COL_1^*(y) = COL(x_1, x_3, y)]$$

$$(\forall y \in V_1 - \{x_3\})[COL_2^*(y) = COL(x_2, x_3, y)]$$

Let V_2 be a 1-homogeneous for COL_1^* of size $\frac{|V_1|-1}{2}$. Let $COL^{**}(x_1, x_3)$ be the color of V_2. Restrict COL_2^* to elements of V_2, though still call it COL_2^*. We reuse the variable name V_2 to be a 1-homogeneous for COL_2^* of size at least $\frac{|V_2|}{2}$. Let $COL^{**}(x_1, x_3)$ be the color of V_2. Let x_4 be the least element of V_2. Repeat the process.

We describe the construction formally.

CONSTRUCTION

$$\begin{aligned} x_1 &= \ 1 \\ V_1 &= \ [n] - \{x_1\} \end{aligned}$$

Let $2 \le i \le R(2, k-1) + 1$. Assume that $x_1, \ldots, x_{i-1}, V_{i-1}$, and $COL^{**} : \binom{\{x_1, \ldots, x_{i-1}\}}{2} \rightarrow \{\text{RED}, \text{BLUE}\}$ are defined.

$$\begin{aligned} x_i &= \ \text{the least element of } V_{i-1} \\ V_i &= \ V_{i-1} - \{x_i\} \text{ (We will change this set without changing its name).} \end{aligned}$$

We define $COL^{**}(x_1, x_i)$, $COL^{**}(x_2, x_i)$, $\ldots$, $COL^{**}(x_{i-1}, x_i)$. We will also define smaller and smaller sets V_i. We will keep the variable name V_i throughout.

For $j = 1$ to $i - 1$

1. $COL^* : V_i \rightarrow \{\text{RED}, \text{BLUE}\}$ is defined by $COL^*(y) = COL(x_j, x_i, y)$.

2. Let V_i be redefined as the largest 1-homogeneous set for COL^*. Note that $|V_i|$ decreases by at most half.

3. $COL^{**}(x_j, x_i)$ is the color of V_i.

KEY: For all $1 \leq i_1 < i_2 \leq i$, for all $y \in V_i$, $COL(x_{i_1}, x_{i_2}, y) = COL^{**}(x_{i_1}, x_{i_2})$.

END OF CONSTRUCTION

When we derive upper bounds on n we will show that the the construction can be carried out for $R(2, k-1) + 1$ stages. For now assume the construction ends.

We have vertices

$$X = \{x_1, x_2, \ldots, x_{R(2,k-1)+1}\}$$

and a 2-coloring COL^{**} of $\binom{X}{2}$. By the definition of $R(2, k-1) + 1$ there exists a set

$$H = \{x_{i_1}, \ldots, x_{i_k}\}.$$

such that the first $k - 1$ elements of it are a 2-homogenous set for COL^{**}. Let the color of this 2-homogenous set be RED. We show that H (including x_{i_k}) is a 3-homogenous set for COL. For notational convenience we show that $COL(x_{i_1}, x_{i_2}, x_{i_3}) = \text{RED}$. The proof for any 3-set of H is similar.

By the definition of COL^{**} for all $y \in V_{i_2}$, $COL(x_{i_1}, x_{i_2}, y) = COL^{**}(x_{i_1}, x_{i_2}) = \text{RED}$. In particular $COL(x_{i_1}, x_{i_2}, x_{i_3}) = \text{RED}$.

We now see how large n must be so that the construction be carried out. Note that in stage i $|V_i|$ be decreases by at most half, i times. Hence $|V_{i+1}| \geq \frac{|V_i|}{2^i}$.

Therefore

$$|V_i| \geq \frac{|V_1|}{2^{1+2+\cdots+(i-1)}} \geq \frac{n-1}{2^{(i-1)^2}}.$$

We want $|V_{R(2,k-1)+1}| \geq 1$. It suffice so take $n = 2^{R(2,k-1)^2} + 1$.

By Theorem 2.9

$$R(2,k-1)^2 + 1 \leq (2^{2k-0.5\lg(k-2)})^2 \leq 2^{4k-\lg(k-2)}.$$

Hence

$$R(3,k) \leq 2^{2^{4k-\lg(k-2)}}.$$

▮

Note 5.2 A slightly better upper bound for $R(3,k)$ can be obtained by using Conlon's upper bound on $R(2,k)$ given in Note 2.10.

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