Homework 2, Morally Due Tue Feb 13, 2018

COURSE WEBSITE: http://www.cs.umd.edu/˜gasarch/858/S18.html
(The symbol before gasarch is a tilde.)

HOMEWORK IS TWO PAGES!!!!!!!!!!!!!

1. (0 points) What is your name? Write it clearly. Staple your HW. When is the midterm tentatively scheduled (give Date and Time)? If you cannot make it in that day/time see me ASAP.

2. (40 points) The infinite Ramsey Theorems we have proven only cared about the CARDINALITY of the homog set (it had to be infinite) and not how it was ordered. In this problem we will look at coloring $\binom{\mathsf{Z}}{a}$ and and $\binom{\mathsf{Q}}{a}$ and see if we can get a homog set with a proscribed order type.

   Definition: A set has H has *order type* Z if there is an order preserving bijection from H to Z. A set has H has *order type* Q if there is an order preserving bijection from H to Q. You can replace Z and Q with any linear orderings.

   (a) (13 points) Prove of Disprove: For all 2-colorings of $\binom{\mathsf{Z}}{1}$ there exists a homog set of order type Z. (In more normal words: for all 2-colorings of Z there is a monochromatic subset that is infinite in both directions.)

   (b) (13 points) Prove of Disprove: For all 2-colorings of $\binom{\mathsf{Q}}{1}$ there exists a homog set of order type Q. (You may use that a countable dense set with no endpoints is of order the Q.)

   (c) (14 points) Prove of Disprove: For all 2-colorings of $\binom{\mathsf{Z}}{2}$ there exists a homog set of order type Z.

   (d) Challenge Question (Bill and Erik thought this was hard until Erik solved it. Now they think it's easy.) Prove of Disprove: For all 2-colorings of $\binom{\mathsf{Q}}{2}$ there exists a homog set of order type Q. (You may use that a countable dense set with no endpoints is of order the Q.)

   SOLUTION TO PROBLEM TWO

   a) FALSE. Counterexample:

Color $\mathsf{Z}^{\ge 0}$ RED

Color $\mathsf{Z}^{<0}$ BLUE

Let H be a homog set. In order for H to have order type Z it needs to have an infinite number of numbers from $\mathsf{Z}^{\ge 0}$ and an infinite number of numbers from $\mathsf{Z}^{<0}$.

A number from $\mathsf{Z}^{\ge 0}$ is RED A number from $\mathsf{Z}^{<0}$ is BLUE

so NOT homog.

b) TRUE.

Let COL be a 2-coloring of Q. Look at

$A = \{q : COL(q) = RED\}$

If A is dense and has no endpoints then we are done. If not then there are two possibilities:

- A is not dense. Then there are two points a, b that are RED but all the points in between them are BLUE. Then $H = (a, b)$ is our homog set of order type Q.
- A is dense, has a left endpoint L, has NO right endpoint. Let

$$H = \{x : x \in A \wedge x > L\}$$

  This set is dense and has no endpoints, so it is of order type Q.
- A is dense, has a right endpoint R, has NO left endpoint. Similar.
- A is dense, has a left endpoint L and a right endpoint R. Similar.

c) FALSE. Counterexample:

Color $\binom{\mathsf{Z}^{\ge 0}}{2}$ RED

Color $\binom{\mathsf{Z}^{<0}}{2}$ BLUE

Color all pairs of numbers with one from $\mathsf{Z}^{\ge 0}$ and one from $\mathsf{Z}^{<0}$ RED.

Let H be a homog set. In order for H to have order type Z it needs to have an infinite number of numbers from $\mathsf{Z}^{\ge 0}$ and an infinite number of numbers from $\mathsf{Z}^{<0}$.

A pair from $\mathsf{Z}^{\ge 0}$ is RED A pair from $\mathsf{Z}^{<0}$ is BLUE so NOT homog.

d) FALSE: Counterexample:

Let $q_1, q_2, q_3, \ldots$ be some enumeration of Q.

Note that there are now TWO orderings on the Q.

- The usual ordering which we denote $<$
- The ordering $q_1 \preceq q_2 \preceq \cdots$. This is not a nice ordering in any way.

For $i < j$ we color (q_i, q_j) RED if $q_i < q_j$ (so if $\preceq$ and $<$ agree), and BLUE otherwise.

If there is a homog RED set or order type Q then after renumbering let it be

$$q_1 < q_2 < q_3 < \cdots$$

Then the map i goes to q_i is an order preserving mapping from N into a set of ordering type Q which is impossible.

If there is a homog RED set or order type Q then after renumbering let it be

$$q_1 > q_2 > q_3 > \cdots$$

Then the map i goes to q_i is an order preserving mapping from N^R into a set of ordering type Q which is impossible.

END OF SOLUTION TO PROBLEM TWO
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3. (60 points)

Recall the proof of the canonical Ramsey theorem using a 16-coloring and the 4-ary Ramsey theorem

(https://www.cs.umd.edu/users/gasarch/COURSES/858/S18/infcanramseytalk.pdf).

(a) (15 points) Show that given an infinite homogenous set H of color 4 (under the 16-coloring), there is either an infinite homogenous set or an infinite min-homogenous set (under the original coloring). Recall that $COL'(x_1 < x_2 < x_3 < x_4) = 4$ if $COL(x_2, x_3) = COL(x_2, x_4)$.

(b) (15 points) Show that given an infinite homogenous set H of color 6 (under the 16-coloring), there is either an infinite homogenous set or an infinite max-homogenous set (under the original coloring). Recall that $COL'(x_1 < x_2 < x_3 < x_4) = 6$ if $COL(x_1, x_4) = COL(x_2, x_4)$.

(c) (15 points) Show that given an infinite homogenous set H of color 11 (under the 16-coloring), there is an infinite homogenous, min-homogenous, or max-homogenous set (under the original coloring). Recall that $COL'(x_1 < x_2 < x_3 < x_4) = 11$ if $COL(x_1, x_2) = COL(x_3, x_4)$.

(d) (15 points) Show that given an infinite homogenous set H of color 12 (under the 16-coloring), there is an infinite homogenous set (under the original coloring). Recall that $COL'(x_1 < x_2 < x_3 < x_4) = 12$ if $COL(x_1, x_3) = COL(x_2, x_4)$.

SOLUTION TO PROBLEM THREE

(a) Let x_1 be the least element of H. Note that for all $x \in H$ with $x > x_1$, $COL(x, y)$ is the same for all $y > x$. Define $COL''(x)$ to be this color. Then, under COL'' there is either an infinite homogenous or rainbow subset of H
$\{x_1\}$. If there is a homogenous subset, then that is also a homogenous set under the original coloring. If there is a rainbow subset, then that is also a min-homogenous subset under the original coloring.

(b) If $H = \{x_1 < x_2 < x_3 < \dots\}$ $H' \subseteq H$ to be the set of all even-indexed elements of H. Then for each $x_{2k} \in H'$, $COL(x_{2i}, x_{2k}) =$

$COL(x_{2k-1}, x_{2k})$ for all $i < k$. So define $COL''(x_{2k}) = COL(x_{2k-1}, x_{2k})$. Then there is either an infinite homogenous or rainbow subset of H' under COL''. If there is an infinite homogenous set, then that is also a homogenous set under the original coloring. If there is a rainbow set, then that is also a max-homogenous set under the original coloring.

(c) Note that if x_1, x_2 are the least elements of H, then $COL(x, y) = COL(x_1, x_2)$ for all $x, y > x_2$. Thus H
x_1, x_2 is a homogenous set under the original coloring.

(d) Let $H = \{x_1 < x_2 < x_3 < \ldots\}$. Define A and B to be the subsets of H consisting only of odd or even indexed elements, respectively. Note that $COL(x_2, x_{2k}) = COL(x_1, x_3)$ for all $k > 1$. Thus $COL(x_{2j-1}, x_{2k-1}) = COL(x_2, x_{2j}) = COL(x_1, x_3)$ for all $j < k$. Thus A is a homogenous set under the original coloring.