Homework 4, Morally Due Tue Feb 27, 2018

COURSE WEBSITE: `http://www.cs.umd.edu/~gasarch/858/S18.html`

1. (0 points) What is your name? Write it clearly. Staple your HW. What type of midterm will there be?

2. (25 points) In class we showed that for all n, $R(n)$ exists. The proof DID NOT give any bounds on $R(n)$. Use a similar proof for the following:

   Let $R(a,c,n)$ be such that for all c-colorings of $\binom{[R(a,c,n)]}{a}$ there exists a homogenous set of size n. Show that $R(a,c,n)$ exists.

   SOLUTION OMITTED

3. (25 points) In class we showed that for all n, $LR(n)$ exists. The proof DID NOT give any bounds on $R(n)$. Use a similar proof for the following:

   Let $LR(a,c,n)$ be such that for all c-colorings of $\binom{\{n,n+1,\ldots,LR(a,c,n)\}}{a}$ there exists a large homogenous set.

   Show that $LR(a,c,n)$ exists.

   SOLUTION OMITTED

4. (25 points) Prove the following using some Can Ramsey Theorem: (Countable means infinite - some books disagree but they are wrong.)

   If $X \subseteq \mathsf{R}^3$ is a countable set of points, no four on the same plane, there exists countable $Y \subseteq X$ such that every 4-subset of Y yields a different volume.

   SOLUTION

   The usual: $COL(p_1, p_2, p_3, p_4)$ is the volume of the shape they create. Apply infinite 4-ary Can Ramsey.

   Let the infinite I-homog set be, after renumbering $\{p_1, p_2, \ldots, \}$.

   We show that of the 16 possible types of homog, only rainbow happens.

   Case 1: If $I = \emptyset$ or I contains 1 or 2 or 3:

   Look at $COL(p_1, p_2, p_3, p_i)$. It is always the same. Hence, for all $4 \leq i \leq 10$ the volumes

$$VOL(p_1, p_2, p_3, p_i)$$

are all of the same. All of those p_i's are on one of two planes, the same height from the plane of p_1, p_2, p_3. Hence since there are 7 of them, 4 must be on the same plane.

Case 2: If $I = \{4\}$ Hence, for $1 \le i \le 7$,

$$VOL(p_i, p_8, p_9, p_{10})$$

all have the same volume. Then make a similar argument.

So all that is left is rainbow!

5. (25 points) (For his problem assume that there is NO cardinality between countable and the cardinality of the reals.) We say $|X| = |\mathsf{R}|$ to mean that X and R are the same size, so there is a bijection between them.

   Prove the following using a Maximal Set argument:

   If $X \subseteq \mathsf{R}^3$, $|X| = |\mathsf{R}|$, no four on the same plane, there exists $Y \subseteq \mathsf{R}^3$, $|Y| = |\mathsf{R}|$, such that every 4-subset of Y yields a different volume.

   SOLUTION

   Let M be a maximal rainbow set. So this means that (1) all 4-sets of M have diff volumes, and (2) for all $p \in X - M$, there are two 4-sets in $M \cup \{p\}$ that have the same volume.

   We map every $p \in X - M$ to the REASON that its is not in M. For every $p \in X - M$ one of the following occurs

   - There exists $\{p_1, p_2, p_3\} \in \binom{M}{3}$ and $\{q_1, q_2, q_3, q_4\} \in \binom{M}{4}$ such that $VOL(p_1, p_2, p_3, p) = VOL(q_1, q_2, q_3, q_4)$. Then map p to $\{p_1, p_2, p_3\} \times \{q_1, q_2, q_3, q_4\}$.
   - There exists $\{p_1, p_2, p_3\} \in \binom{M}{3}$ and $\{q_1, q_2, q_3\} \in \binom{M}{3}$ such that $VOL(p_1, p_2, p_3, p) = VOL(q_1, q_2, q_3, p)$. Then map p to $\{p_1, p_2, p_3\} \times \{q_1, q_2, q_3\}$.

   Call the above map f.

   $f^{-1}(\{p_1, p_2, p_3\}, \{q_1, q_2, q_3, q_4\})$ is the set of points that, together with p_1, p_2, p_3, form a solid of volume $VOL(q_1, q_2, q_3, q_4)$. The set of such points is one of two planes, so there are at most 6 such points to avoid 4 on a plane.

$f^{-1}(\{p_1, p_2, p_3\} \times \{q_1, q_2, q_3\})$: there is a number r such that for any element p in this set, if h_1 is the distance from the p_1, p_2, p_3-plane to p and h_2 is the distance from the q_1, q_2, q_3-plane to p, then $h_1 = rh_2$. Hence the inverse image is a plane, so there are at most 3 such points to avoid 4 on a plane.

So the function f is finite-to-1.

SO, there is a finite-to-1 function from $X - M$ to M. Therefore M is uncountable.

6. (0 points but you must do this so we can discuss) On the course website is a link to a review of a book on the Banach-Tarski Pardox. Read the review. Be prepared to discuss if you think the BT paradox is TRUE or FALSE or SOMETHING ELSE. There is no right answer here but I want to know what you think.