Take Home Midterm. Given out Feb 27
Morally Due THURSDAY March 8. Sick Cat Day-TUESDAY March 13
FIVE PAGES!!!!!!!!!!!!!!!

1. (0 points) What is your name? Write it clearly. Staple this.

2. (25 points) Find a function \( f(n) \) such that the following is true, and prove it:
   - For any coloring (any number of colors) of \( \{1, \ldots, f(n)\} \) there exists either \( n \) elements that are the same color OR there exists \( n \) elements that are all different colors.
   - There exists a coloring (any number of colors) of \( \{1, \ldots, f(n) - 1\} \) with neither \( n \) elements that are the same color NOR with \( n \) elements that are all different colors.

3. (25 points)
   (a) Find a function \( f(n) \) such that the following is true, and prove it using a maximal-set argument.
   \( \text{If } X \text{ is a set of points in the plane, no three colinear, of size } f(n) \text{ then there exists } Y \subseteq X \text{ of size } n \text{ such that no four points form a trapezoid.} \)

   (b) Find a function \( f(n, k) \) such that the following is true, and prove it using a maximal-set argument. (We assume \( n, k \geq 3 \).)
   \( \text{If } X \text{ is a set of points in the plane, no } k \text{ colinear, of size } f(n, k) \text{ then there exists } Y \subseteq X \text{ of size } n \text{ such that no four points form a trapezoid.} \)
4. (25 points) Let $COL$ be a coloring of $\mathbb{N} \times \mathbb{N}$. A mono grid is a pair of sets $A, B \subseteq \mathbb{N}$ such that the $COL$ restricted to $A \times B$ is monochromatic. If $A$ and $B$ are both of size infinite we say its an infinite mono grid of size $n$. If $A$ and $B$ are both of size $n$ we say its a mono grid of size $n$.

(a) Prove or disprove: For all 2-colorings of $\mathbb{N} \times \mathbb{N}$ there exists an infinite mono grid.

(b) Find a function $f(n)$ such that the following is true (and prove it), or show that no such function exists:

For all 2-colorings of $[f(n)] \times [f(n)]$ there exists a mono grid of size $n$.

(c) Find a function $f(n, c)$ such that the following is true (and prove it), or show that no such function exists:

For all $c$-colorings of $[f(n, c)] \times [f(n, c)]$ there exists a mono grid of size $n$.

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(50 points) In this problem we guide you through a finite version of Mileti’s proof of the infinite can Ramsey Theorem. We work backwards by taking the last part of the proof first.

**ADVICE:** (1) When the infinite proof asked for an INFINITE subset, here instead take a subset that is of size square root of what we had, (2) make gross overestimates to get this all to work – trying to refine it gets complicated.

**PROBLEM MILLONE**

Find a function \( f(n) \) such that the following lemma holds.

**Lemma** Let \( \text{COL} \) be an \( \omega \)-coloring of \( (\lceil f(n) \rceil)^2 \). Assume that

- For all \( 1 \leq i \leq f(n) - 2 \), for all \( i < k_1 < k_2 \leq f(n) \)
  \[
  \text{COL}(i, k_1) \neq \text{COL}(i, k_2).
  \]

- For all \( 1 \leq i < j \leq f(n) - 1 \), for all \( k \geq j + 1 \),
  \[
  \text{COL}(i, k) \neq \text{COL}(j, k).
  \]

Then there exists a rainbow set of size \( n \). (Note that we DO NOT have one yet since \( \text{COL}(3, 8) = \text{COL}(4, 11) \) is possible.)

**PROBLEM MILLTWO** Find a function \( g(n) \) such that the following lemma is true: **Lemma** Let \( \text{COL}' \) be a coloring of \( [g(n)] \) where the colors are of the form \((H, c)\) and \((RB, i)\). Then one of the following must occur:

(a) There exists \( c \) and \( Y \subseteq [g(n)] \), \(|Y| \geq n\), such that every element of \( Y \) is colored \((H, c)\).

(b) There exists \( Y \subseteq [g(n)] \), \(|Y| \geq n\), such that every element of \( Y \) is colored \((H, *)\) and they all have different second components.

(c) There exists \( i \) and \( Y \subseteq [g(n)] \), \(|Y| \geq n\), such that every element of \( Y \) is colored \((RB, i)\).

(d) There exists \( Y \subseteq [g(n)] \), \(|Y| \geq n\), such that every element of \( Y \) is colored \((RB, *)\) and they all have different second components.

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PROBLEM MILLTHREE
Find a function $h(n)$ such that the following lemma is true: Lemma
Let $COL$ be an $\omega$-coloring of $\binom{[h(n)]}{2}$ Assume there is a coloring $COL'$ of $[h(n)]$ where the colors are of the form $(H, c)$ and $(RB, i)$, and the following holds:

- If $COL'(x) = (H, c)$ then for all $z > x$ $COL(x, z) = c$.
- If $COL'(x) = (RB, i)$ then for all $z_1 \neq z_2 > x$, $COL(x, z_1) \neq COL(x, z_2)$.
- If $COL'(x) = (RB, i)$ and $COL'(y) = (RB, i)$ then for all $z > \max\{x, y\}$, $COL(x, z) = COL(y, z)$.
- If $COL'(x) = (RB, i)$ and $COL'(y) = (RB, j)$ (with $i \neq j$) then for all $z > \max\{x, y\}$, $COL(x, z) \neq COL(y, z)$.

Then one of the followings holds:
(a) There is a homog set of size $n$.
(b) There is a min-homog set of size $n$.
(c) There is a max-homog set of size $n$.
(d) There is a rainbow set of size $n$.

PROBLEM MILLFOUR
Find a function $BILL(n)$ (sorry, I’m running out of letters) such that the following lemma is true: Lemma: Let $COL$ be a $\omega$-coloring of $\binom{[BILL(n)]}{2}$ Then there is a subset of $[BILL(n)]$ of size $n$ and a coloring $COL'$ of that subset, where the colors are of the form $(H, c)$ and $(RB, i)$, such that the following holds:

- If $COL'(x) = (H, c)$, then for all $z > x$, $COL(x, z) = c$.
- If $COL'(x) = (RB, i)$, then for all $z_1, z_2 > x$, $COL(x, z_1) \neq COL(x, z_2)$.
- If $COL'(x) = (RB, i)$ and $COL'(y) = (RB, i)$, then for all $z > \max\{x, y\}$, $COL(x, z) = COL(y, z)$.
- If $COL'(x) = (RB, i)$ and $COL'(y) = (RB, j)$ (with $i \neq j$), then for all $z > \max\{x, y\}$, $COL(x, z) \neq COL(y, z)$.

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PROBLEM MILLFIVE Put all of this together to (easily) find a function $CR(n)$ (for Can Ramsey) such that the following theorem is true:

Theorem Let $COL$ be an $\omega$-coloring of $\binom{|CR(n)|}{2}$. Then one of the following holds:

(a) There is a homog set of size $n$.
(b) There is a min-homog set of size $n$.
(c) There is a max-homog set of size $n$.
(d) There is a rainbow set of size $n$.

6. (25 points) (This is a NEW problem – nothing to do with Finite Can Ramsey.) Let $(L, \preceq)$ be a well quasi order. Let $2^{\text{fin}L}$ be the set of FINITE subsets of $L$. We DEFINE an order $\preceq'$ on $2^{\text{fin}L}$:

$A \preceq' B$ if there is an injection $f$ from $A$ to $B$ such that $x \preceq f(x)$.

($\emptyset \preceq' B$ is always true: use the empty function and the condition holds vacuously.)

Show that $(2^{\text{fin}L}, \preceq')$ is a well quasi order.

(NOTE- this proof will use that wqo are closed under cross product, but the proof I have does not use Ramsey Theory directly.)