Take Home Midterm. Given out Feb 27
Morally Due THURSDAY March 8. Sick Cat Day-TUESDAY March 13
FIVE PAGES!!!!!!!!!!!!!!!

1. (0 points) What is your name? Write it clearly. Staple this.

2. (25 points) Find a function $f(n)$ such that the following is true, and prove it:

   - For any coloring (any number of colors) of $\{1, \ldots, f(n)\}$ there exists either n elements that are the same color OR there exists n elements that are all different colors.
   - There exists a coloring (any number of colors) of $\{1, \ldots, f(n)-1\}$ with neither n elements that are the same color NOR with n elements that are all different colors.

   Solution:

   Let $f(n) = (n-1)^2 + 1$. Suppose that you color $(n-1)^2 + 1$ elements with $\le n-1$ colors. Then the average number of times each color is used must be greater than or equal to $\frac{(n-1)^2+1}{n-1} > n-1$. Thus there is a color used at least n times. If there are only $(n-1)^2$ elements, we can simply use each of $n-1$ colors $n-1$ times, and there will be no homogenous or rainbow sets of size n.

3. (25 points)

   (a) Find a function $f(n)$ such that the following is true, and prove it using a maximal-set argument.

   If X is a set of points in the plane, no three colinear, of size $f(n)$ then there exists $Y \subseteq X$ of size n such that no four points form a trapezoid.

   (b) Find a function $f(n,k)$ such that the following is true, and prove it using a maximal-set argument. (We assume $n, k \ge 3$.)

   If X is a set of points in the plane, no k colinear, of size $f(n,k)$ then there exists $Y \subseteq X$ of size n such that no four points form a trapezoid.

Solution:

Let $f(n) = (n-1)\binom{n-1}{2} + n$, and consider a maximal trapezoid free set $Y \subseteq X$. Suppose $|Y| \leq n-1$. Let $x \in X \setminus Y$. Since Y is maximal, there exists $\{a, (b, c)\} \in Y \times \binom{Y}{2}$ such that $\{x, a, b, c\}$ forms a trapezoid with parallels lines ax and bc. We think of $\{a, (b, c)\}$ as the reason $x \notin Y$. Define $\varphi : X \setminus Y \to Y \times \binom{Y}{2}$ by $\varphi(x) = (a, (b, c))$ as above. If there are multiple possible choices for $\varphi(x)$, choose arbitrarily. Then if $\varphi(x_1) = \varphi(x_2) = (a, (b, c))$ then the line $x_1 a$ and $x_2 a$ are both parallel to bc, hence a, x_1, x_2 are colinear which violates the hypothesis. Hence φ is injective. Therefore the domain has size $\leq$ the codomain, so

$$|X - Y| \leq |Y| \times \binom{|Y|}{2}.$$

$$|X| \leq |Y| \times \binom{|Y|}{2} + |Y| \leq (n-1) \times \binom{(n-1)}{2} + n - 1.$$

This contradicts the size of $|X|$.

Part 2 we leave to you.

NOTE: Many students mapped x to $\{a, b, c\}$ rather than $(a, (b, c))$. This can work but is trickier and most students formally got it wrong though we did not penalize.

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4. (25 points) Let COL be a coloring of $\mathsf{N} \times \mathsf{N}$. A *mono grid* is a pair of sets $A, B \subseteq \mathsf{N}$ such that the COL restricted to $A \times B$ is monochromatic. If A and B are both of size infinite we say its an *infinite mono grid of size n.* If A and B are both of size n we say its an *mono grid of size n.*

   (a) Prove or disprove: For all 2-colorings of $\mathsf{N} \times \mathsf{N}$ there exists an infinite mono grid.

   (b) Find a function $f(n)$ such that the following is true (and prove it), or show that no such function exists:
   For all 2-colorings of $[f(n)] \times [f(n)]$ *there exists a mono grid of size n.*

   (c) Find a function $f(n, c)$ such that the following is true (and prove it), or show that no such function exists:
   For all c-colorings of $[f(n, c)] \times [f(n, c)]$ *there exists a mono grid of size n.*

Solution:

(a) False. Consider the coloring $C(a, b) = 1$ if $a < b$, and $C(a, b) = 2$ otherwise.

(b) We give several proofs:

PROOF ONE: $f(n) = 2^{2n+1}$

Let COL be a 2-coloring of $[f(n)] \times [f(n)]$.

View COL as a 2^n-coloring of the rows. There are 2^{2n+1} rows so by PHP there are $\frac{2^{2n+1}}{2^n} = 2^{n+1}$ rows that are the same color (we just need 2^n). Look just at those rows. Assume the color has at least half R's. Each row is 2^{n+1} long, so there is a set of 2^n positions where it is R. Take the 2^n rows, the 2^n positions where they are R and you have your mono grid.

PROOF TWO: $f(n) = 2^{3n+1}$.

Look at ROW 1: Let c_1 be the majority color KILL ALL THOSE WHO DISAGREE. There are now 2^{3n} columns.

Look at ROW 2: Let c_2 be the majority color KILL ALL THOSE WHO DISAGREE. There are now 2^{3n-1} columns.

Keep doing this. until

Look at ROW $2n+1$: Let c_{2n+1} be the majority color KILL.... There are now $2^{3n-2n} = 2^n$ columns.

We have $c_1, \ldots, c_{2n+1}$. There are n that are the same color. Take those rows.

PROOF THREE: $f(n) = 2^{2n}$ suffices. Let $g(a,b)$ be equal to the number of elements required to find either a mono grid for color 1 of size a or a mono grid for color 2 of size b. Then we can construct a mono grid of size n given sufficiently many points as follows: Consider point 1. If there are $g(n-1,n)$ elements b of B such that $COL(1,b) = 1$, let $COL'(1) = 1$ and let B_1 be the subset of N such that $COL(1,b) = 1$ for all $b \in B_1$. Otherwise, there will be $g(n,n-1)$ elements b in B such that $COL(1,b) = 2$. In this case $COL'(1) = 2$ and B_1 is the set of things such that $COL(1,b) = 2$. Suppose the former case occurred. Then either there is a mono grid $A \times B$ for color 1 of size $(n-1)$ such that $1 \notin A$ and $B \subseteq B_1$, or there is a mono grid $A \times B$ for color 2 of size n such that $1 \notin A$ and $B \subseteq B_1$. Either way, we can then construct a mono grid of size n on the original grid.

A similar argument shows $g(a,b) \le g(a-1,b) + g(a,b-1)$. We will now show $g(a,b) \le 2^{a+b}$. First, it is easy to check $g(x,1) = g(1,x) = x \le 2^{x+1}$. Now, for $a,b > 1$ we have $g(a,b) \le g(a-1,b) + g(a,b-1) \le 2 \cdot 2^{a+b-1} = 2^{a+b}$, as desired. Since $g(n,n) \le 2^{2n}$, $f(n) = 2^{2n}$ suffices.

(c) $f(n) = c^{cn}$ suffices. Use the previous argument, except instead of $g(a,b) \le g(a-1,b) + g(a,b-1)$ show that $g_c(a_1, a_2, \ldots, a_c) \le g_c(a_1 - 1, a_2, \ldots, a_c) + g_c(a_1, a_2 - 1, \ldots, a_c) + \cdots + g_c(a_1, a_2, \ldots, a_c - 1)$. Check that $g_c(1, a_2, a_3, \ldots, a_c) \le g_{c-1}(a_2, a_3, \ldots, a_c) \le (c-1)^{(a_2+a_3+\cdots+a_c)} \le c^{(1+a_2+a_3+\cdots+a_c)}$. Then by induction $g_c(a_1, a_2, \ldots, a_c) \le c^{(a_1+a_2+\cdots+a_n)}$. Therefore $g_c(n, n, \ldots, n) \le c^{cn}$, so $f(n) = c^{cn}$ suffices.

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5. (50 points) In this problem we guide you through a finite version of Mileti's proof of the infinite can Ramsey Theorem. We work backwards by taking the last part of the proof first.

ADVICE: (1) When the infinite proof asked for an INFINITE subset, here instead take a subset that is of size square root of what we had, (2) make gross overestimates to get this all to work – trying to refine it gets complicated.

PROBLEM MILLONE

Find a function $f(n)$ such that the following lemma holds.

Lemma Let COL be an ω-coloring of $\binom{[f(n)]}{2}$. Assume that

- For all $1 \le i \le f(n) - 2$, for all $i < k_1 < k_2 \le f(n)$
$$COL(i, k_1) \ne COL(i, k_2).$$
- For all $1 \le i < j \le f(n) - 1$, for all $k \ge j + 1$,
$$COL(i, k) \ne COL(j, k).$$

Then there exists a rainbow set of size n. (Note that we DO NOT have one yet since $COL(3, 8) = COL(4, 11)$ is possible.)

SOLUTION TO MILLONE

We will pick f later. We define a sequence of z's and a sequence of H's

$z_1 = 1$

$H_1 = \{2, 3, \ldots, f(n)\}$.

Assume that $z_1, \ldots, z_i$ have been chosen and that all of the edges between them are different colors. Let $SETCOL_i$ be the set of colors of edges (there are $\binom{i}{2}$ of them). All of the elements of H_i are $> z_i$. Find the least element z of H_i such that,

$$(\forall 1 \le j \le i)[COL(z_j, z) \notin SETCOL_i].$$

AND

$$(\forall 1 \le j_1 < j_2 \le i)[COL(z_{j_1}, z) \ne COL(z_{j_2}, z)].$$

FIRST KEY: The second clause holds for all z

SECOND KEY: we need to show that there exists a z satisfying the first clause. We claim that such a z exists within the first i^3 elements of H_i. Assume, by contradiction, that there is no such z. We map each $z \in H_i$ to the REASON it does not work. Map H_i to $\{1, \ldots, i\} \times SETCOL_i$ as follows:

$z \in H_i$. z DID NOT get to be z_{i+1}. Hence there is some j (take the least one) such that $COL(z_j, z) = c \in SETCOL_i$. Let j be the least such j. Map z to (j, c).

Restrict this map to the first i^3 elements of H_{i-1}. Now it maps i^3 elements to $i \times \binom{i}{2}$ elements, which is $< i^3$. Hence there is z, z' within the first i^3 elements of H_i such that there is a j with $COL(z_j, z) = c$ and $COL(z_j, z') = c$ This violated $COL(z_j)$ only has one color coming ot of it.

We now define

z_{j+1} is the z found

H_{j+1} is H_j MINUS all the elements in H_j that were less than z that did not make it. So sad for them :-(.

Since $|H_{i+1}| \ge |H_i| - i^3$ we have

$|H_n| \ge |H_0| - 1^2 - 2^3 - 3^3 - \cdots - n^3 \ge |H_0| - n^4$.

Since we need to do the process n times take $f(n) = n^4$.

END OF SOLUTION TO MILLONE

PROBLEM MILLTWO Find a function $g(n)$ such that the following lemma is true: **Lemma** Let COL' be a coloring of $[g(n)]$ where the colors are of the form (H, c) and (RB, i). Then one of the following must occur:

(a) There exists c and $Y \subseteq [g(n)]$, $|Y| \ge n$, such that every element of Y is colored (H, c).

(b) There exists $Y \subseteq [g(n)]$, $|Y| \ge n$, such that every element of Y is colored $(H, *)$ and they all have different second components.

(c) There exists i and $Y \subseteq [g(n)]$, $|Y| \ge n$, such that every element of Y is colored (RB, i).

(d) There exists $Y \subseteq [g(n)]$, $|Y| \ge n$, such that every element of Y is colored $(RB, *)$ and they all have different second components.

SOLUTION TO PROBLEM MILLTWO

Either $g(n)/2$ of the numbers are colored $(H, *)$ or are colored $(RB, *)$. Assume its $(H, *)$ (the other case is similar).

Of these $g(n)/2$ elements either there exists c such that $\sqrt{g(n)/2}$ are colored (H, c) OR there exists $\sqrt{g(n)/2}$ with different second components.

So we need $\sqrt{g(n)/2} \ge n$. We take $g(n) = 2n^2$.

END OF SOLUTION TO PROBLEM MILLTWO

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PROBLEM MILLTHREE

Find a function $h(n)$ such that the following lemma is true: **Lemma** Let COL be an ω-coloring of $\binom{[h(n)]}{2}$ Assume there is a coloring COL' of $[h(n)]$ where the colors are of the form (H,c) and (RB,i), and the following holds:

- If $COL'(x)=(H,c)$ then for all $z>x$ $COL(x,z)=c$.
- If $COL'(x)=(RB,i)$ then for all $z_1\neq z_2>x$, $COL(x,z_1)\neq COL(x,z_2)$.
- If $COL'(x)=(RB,i)$ and $COL'(y)=(RB,i)$ then for all $z>\max\{x,y\}$, $COL(x,z)=COL(y,z)$.
- If $COL'(x)=(RB,i)$ and $COL'(y)=(RB,j)$ (with $i\neq j$) then for all $z>\max\{x,y\}$, $COL(x,z)\neq COL(y,z)$.

Then one of the followings holds:

(a) There is a homog set of size n.

(b) There is a min-homog set of size n.

(c) There is a max-homog set of size n.

(d) There is a rainbow set of size n.

SOLUTION TO MILLTHREE

By the solution to MILLTWO one of the following holds:

(a) There are $\sqrt{h(n)/2}$ with (H,c). Then there is a homog set of size $\sqrt{h(n)/2}$ so we need $h(n)\geq 2n^2$.

(b) There are $\sqrt{h(n)/2}$ with $(H,*)$, all different second parts. Then there is a min-homog set of size $\sqrt{h(n)/2}$ so we need $h(n)\geq 2n^2$.

(c) There are $\sqrt{h(n)/2}$ with (RB,i) Then there is a max-homog set of size $\sqrt{h(n)/2}$ so we need $h(n)\geq 2n^2$.

(d) There are $\sqrt{h(n)/2}$ with $(RB,*)$, all different second parts. You DO NOT have a rainbow set! You use the solution to MILLONE to get a rainbow set of size $(\sqrt{h(n)/2})^{1/4}=(h(n)/2)^{1/8}$.

So take $h(n)=2n^8$.

END OF SOLUTION TO MILLTHREE

PROBLEM MILLFOUR

Find a function $BILL(n)$ (sorry, I'm running out of letters) such that the following lemma is true: **Lemma:** Let COL be a ω-coloring of $\binom{[BILL(n)]}{2}$ Then there is a subset of $[BILL(n)]$ of size n and a coloring COL' of that subset, where the colors are of the form (H, c) and (RB, i), such that the following holds:

- If $COL'(x) = (H, c)$, then for all $z > x$, $COL(x, z) = c$.
- If $COL'(x) = (RB, i)$, then for all $z_1, z_2 > x$, $COL(x, z_1) \neq COL(x, z_2)$.
- If $COL'(x) = (RB, i)$ and $COL'(y) = (RB, i)$, then for all $z > \max\{x, y\}$, $COL(x, z) = COL(y, z)$.
- If $COL'(x) = (RB, i)$ and $COL'(y) = (RB, j)$ (with $i \neq j$), then for all $z > \max\{x, y\}$, $COL(x, z) \neq COL(y, z)$.

SOLUTION TO MILLFOUR

$$\begin{aligned} V_0 = &\ [BILL(n)] \\ x_1 = &\ 1 \end{aligned}$$

If $(\exists c)|\{v \in V_0 \mid COL(x_1, v) = c\}| \geq \sqrt{|BILL(n)|}$ then:

- $c_1 = (H, c)$
- $V_1 = \{v \in V_0 \mid COL(x_1, v) = c\}$. (Note that $|V_1| \geq \sqrt{|BILL(n)|}$)

If $(\forall c)|\{v \in V_0 \mid COL(x_1, v) = c\}| < \sqrt{|BILL(n)|}$ then:

- $V_1 = \{v \in V_0 \mid (\exists c)[COL(x_1, v) = c \wedge (\forall x_1 < u < v)[COL(x_1, u) \neq c]]\}$ (so v is the first first with $COL(x_1, v) = c$. Hence there will only be ONE v with $COL(x_1, v) = c$.) (Note that $|V_1| \geq \sqrt{|BILL(n)|}$)
- $c_1 = (RB, 1)$. (The 1 marks that this is the first rainbow-color assigned.)

Let $i \geq 2$, and assume that V_{i-1} is defined. We define x_i, c_i, and V_i:

x_i gets the least element of V_{i-1}.

For all colors c let

$Y_c = \{x \in V_{i-1} : COL(x_i, x) = c\}$

Also let:

$Y_\omega = \{x \in V_{i-1} : (\forall y \in V_{i-1}, y < x)[COL(x_i, x) \neq COL(x_i, y)]\}$

(So all colors coming out of x are different.

If there exists c such that $|Y_c| \geq \sqrt{|V_{i-1}|}$ then

$$\begin{aligned} c_i = &\ (H, c) \\ V_i = &\ Y_c \end{aligned}$$

If no such c exist then there exists Y_ω with $|Y_\omega| \geq \sqrt{|V_{i-1}|}$. with all of the vertices coming out of it being different colors. We initially take

$V_i = Y_\omega$

But we may thin it out. And we haven't colored x_i yet.

Do the following:

For all $1 \le j \le i-1$ such that $COL'(x_j) = (RB, k)$ for some k then:

(a) If $|\{y \in Y_\omega : COL(x_j, y) = COL(x_i, y)\}| \ge \sqrt{|V_i|}$ then let V_i be this set and let $c_i = c_j$. (So $COL'(x_i)$ will be of the form (RB, k) for some k). You are done and do not go to the next j.

(b) If $|\{y \in Y_\omega : COL(x_j, y) = COL(x_i, y)\}| < \sqrt{|V_i|}$ then let V_i be the Y_ω minus those vertices.

If Case 1 ever happens then we are done. If Case 2 always happens then note that x_i disagrees with every x_j on every element $> x_i$. We c_i with (RB, k) where k is the least number not used for a rainbow color yet.

END OF CONSTRUCTION

The KEY for us is how big is V_i.

In the worst case we keep on subtracting $\sqrt{|V_i|}$ vertices and then at the very last stage take a square root. Even though $|V_i|$ keeps getting smaller within a stage we won't use this (so our results are not as good as they could be).

Lets start at the beginning

We have V_{i-1}.

We do the Y_ω thing

We now have a set of size $\sqrt{|V_{i-1}|}$.

We then subtract $|V_{i-1}|^{1/4}$ i times.

So we have

$|V_i| \ge \sqrt{|V_i|^{1/2} - i|V_i|^{1/4}}$

To simplify we will assume $i|V_i|^{1/4} \le \frac{|V_i|^{1/2}}{2}$. (we later make sure that all $|V_i| \ge 16i^4$ so this true). Hence

$$|V_i| \ge \sqrt{|V_{i-1}|^{1/2} - i|V_{i-1}|^{1/4}} \ge \sqrt{\frac{|V_{i-1}|^{1/2}}{2}}$$

So we have

$$|V_i| \ge \frac{|V_{i-1}|^{1/4}}{\sqrt{2}}$$

We get really lazy here and make this even easier to deal with by assuming $|V_i| \ge \frac{|V_{i-1}|^{1/4}}{\sqrt{2}} \ge |V_{i-1}|^{1/5}$ (we later make sure that all $|V_i| \ge 4^5$ to make this true).

$$|V_i| \ge |V_{i-1}|^{1/5}$$

$$|V_n| \ge |V_0|^{(1/5)^n}$$

So we need $|V_0|^{(1/5)^n} \ge 4^5$, so $|V_0| \ge 4^{5^{n+1}}$.

We also need $|V_0|^{(1/5)^n} \ge 16n^4$, so $|V_0| \ge (16n^4)^{5^n}$

so we can take $BILL(n) = 4^{5^{n+1}} + (16n^4)^{5^n}$

END OF SOLUTION TO MILLFOUR

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PROBLEM MILLFIVE Put all of this together to (easily) find a function $CR(n)$ (for Can Ramsey) such that the following theorem is true:

Theorem Let COL be an ω-coloring of $\binom{[CR(n)]}{2}$. Then one of the following holds:

(a) There is a homog set of size n.

(b) There is a min-homog set of size n.

(c) There is a max-homog set of size n.

(d) There is a rainbow set of size n.

SOLUTION TO MILLFIVE

Let $CR(n) = BILL(2n^8) = 4^{5^{2n^8+1}} + (256n^{32})^{5^{2n^8}}$.

We leave it to the reader to see that this works.

END OF SOLUTION TO MILLFIVE

6. (25 points) (This is a NEW problem – nothing to do with Finite Can Ramsey.) Let $(L, \preceq)$ be a well quasi order. Let $2^{\text{fin}L}$ be the set of FINITE subsets of L. We DEFINE an order $\preceq'$ on $2^{\text{fin}L}$:

   $A \preceq' B$ if there is an injection f from A to B such that $x \preceq f(x)$.

   ($\emptyset \preceq' B$ is always true: use the empty function and the condition holds vacuously.)

   Show that $(2^{\text{fin}L}, \preceq')$ is a well quasi order.

   (NOTE- this proof will use that wqo are closed under cross product, but the proof I have does not use Ramsey Theory directly.)

   SOLUTION TO PROBLEM SIX

   Throughout 'smallest' means smallest CARDINALITY of a set.

   Assume, BWOC, that $(2^{\text{fin}L}, \preceq')$ is a NOT a wqo.

   Let A_1 be the smallest set that begins a bad sequence.

   Let A_2 be the smallest set that is the second element of a bad sequence that begins with A_1

   For all $i \geq 3$

   Let A_i be the smallest set that is the ith element of a bad sequence that begins with $A_1, A_2, \ldots, A_{i-1}$.

   Note that

   $$A_1, A_2, A_3, \ldots$$

   is a *minimal bad sequence*.

   None of the A_i's can be empty since its a bad sequence.

   Let B_i be A_i minus an element.

   The elements are picked arb, however lets call the set of such elements $MINUS$.

   Let $\mathcal{B} = \{B_1, B_2, \ldots\}$.

   Claim: $\mathcal{B}$ with the order $\preceq'$ is a wqo

   Proof of Claim: Assume, BWOC, that there is a bad sequence:

   $$B_{i_1}, B_{i_2}, \ldots$$

We can assume that i_1 is the smallest index that appears (take the smallest one that appears and start there). Aside from that we DO NOT know anything about the order of the i_j's.

Look at the sequence

$$A_1, A_2, \ldots, A_{i_1-1}, B_{i_1}, B_{i_2}, \ldots$$

(NOTE we DO NOT KNOW, NOR DO WE THINK that $i_1 < i_2 < \cdots$)

We show this is a BAD sequence.

(a) Since $A_1, A_2, \ldots$ is a bad sequence there will be no uptick in the first $i_1 - 1$ elements of the sequence.

(b) Since $B_{i_1}, B_{i_2}, \ldots$ is a bad sequence there will be on uptick in the elements after A_{i_1-1}.

(c) Assume, BWOC, that we have $i < i_j$ and $A_i \preceq' B_{i_j}$. Take the injection from A_i to B_{i_j} and view it as an injection from A_i to A_{i_j}. Hence $i < i_j$ and $A_i \preceq A_{i_j}$. Hence we have an uptick in the BAD SEQUENCE $A_1, A_2, \ldots$. This is a contradiction.

SO

$$A_1, A_2, \ldots, A_{i_1-1}, B_{i_1}, B_{i_2}, \ldots$$

is a bad sequence. Look at its i_1 element. Recall how A_{i_1} was defined:

Let A_{i_1} be the smallest set that is the i_1th element of a bad sequence that begins with $A_1, A_2, \ldots, A_{i_1-1}$.

BUT we are now looking at a bad sequence that begins with

$$A_1, A_2, \ldots, A_{i_1-1}$$

with i_1th element B_{i_1}, and $|B_{i_1}|$ is A_{i_1} with one element missing so it is SMALLER. This is a contradiction.

So $\mathcal{B}$ with $\preceq'$ is a wqo.

End of Proof of Claim

SO $\mathcal{B}$ under $\preceq'$ is a wqo

$MINUS$ under $\preceq$ is a subset of a wqo so its a wqo.

So $\mathcal{B} \times MINUS$ is a wqo.

Look at the original bad sequence (Sounds like a rap singer's nickname *the original badass sequence!* - Maybe he or she could do a rap song about *badass sequences* – it could not be worse than the BW "rap").

SEQONE: $A_1, A_2, \ldots,$

View this as

SEQTWO: $(B_1, b_1), (B_2, b_2), \ldots$

Where $A_i = B_i \cap \{b_i\}$.

Since SEQTWO has an uptick, SEQONE has an uptick.