1. (0 points) What is your name? Write it clearly. Staple this.

2. (25 points) Find a function $f(n)$ such that the following is true, and prove it:

   - For any coloring (any number of colors) of $\{1, \ldots, f(n)\}$ there exists either $n$ elements that are the same color OR there exists $n$ elements that are all different colors.
   - There exists a coloring (any number of colors) of $\{1, \ldots, f(n) - 1\}$ with neither $n$ elements that are the same color NOR with $n$ elements that are all different colors.

   **Solution:**

   Let $f(n) = (n - 1)^2 + 1$. Suppose that you color $(n - 1)^2 + 1$ elements with $\leq n - 1$ colors. Then the average number of times each color is used must be greater than or equal to $\frac{(n-1)^2+1}{n-1} > n - 1$. Thus there is a color used at least $n$ times. If there are only $(n - 1)^2$ elements, we can simply use each of $n - 1$ colors $n - 1$ times, and there will be no homogenous or rainbow sets of size $n$.

3. (25 points)

   (a) Find a function $f(n)$ such that the following is true, and prove it using a maximal-set argument.

   *If $X$ is a set of points in the plane, no three colinear, of size $f(n)$ then there exists $Y \subseteq X$ of size $n$ such that no four points form a trapezoid.*

   (b) Find a function $f(n, k)$ such that the following is true, and prove it using a maximal-set argument. (We assume $n, k \geq 3$.)

   *If $X$ is a set of points in the plane, no $k$ colinear, of size $f(n, k)$ then there exists $Y \subseteq X$ of size $n$ such that no four points form a trapezoid.*
Solution:
Let \( f(n) = (n-1)(n^{-1}) + n \), and consider a maximal trapezoid free set \( Y \subseteq X \). Suppose \( |Y| \leq n - 1 \). Let \( x \in X \setminus Y \). Since \( Y \) is maximal, there exists \( \{a, (b, c)\} \in Y \times \binom{Y}{2} \) such that \( \{x, a, b, c\} \) forms a trapezoid with parallels lines \( ax \) and \( bc \). We think of \( \{a, (b, c)\} \) as the reason \( x \notin Y \). Define \( \varphi : X \setminus Y \to Y \times \binom{Y}{2} \) by \( \varphi(x) = (a, (b, c)) \) as above. If there are multiple possible choices for \( \varphi(x) \), choose arbitrarily. Then if \( \varphi(x_1) = \varphi(x_2) = (a, (b, c)) \) then the line \( x_1a \) and \( x_2a \) are both parallel to \( bc \), hence \( a, x_1, x_2 \) are colinear which violates the hypothesis. Hence \( \varphi \) is injective. Therefore the domain has size \( \leq \) the codomain, so

\[
|X \setminus Y| \leq |Y| \times \binom{|Y|}{2}.
\]

\[
|X| \leq |Y| \times \binom{|Y|}{2} + |Y| \leq (n - 1) \times \binom{n - 1}{2} + n - 1.
\]

This contradicts the size of \( |X| \).

Part 2 we leave to you.

\textbf{NOTE:} Many students mapped \( x \) to \( \{a, b, c\} \) rather than \( (a, (b, c)) \). This can work but is trickier and most students formally got it wrong though we did not penalize.
4. (25 points) Let COL be a coloring of $\mathbb{N} \times \mathbb{N}$. A mono grid is a pair of sets $A, B \subseteq \mathbb{N}$ such that the COL restricted to $A \times B$ is monochromatic. If $A$ and $B$ are both of size infinite we say its an infinite mono grid of size $n$. If $A$ and $B$ are both of size $n$ we say its an mono grid of size $n$.

(a) Prove or disprove: For all 2-colorings of $\mathbb{N} \times \mathbb{N}$ there exists an infinite mono grid.

(b) Find a function $f(n)$ such that the following is true (and prove it), or show that no such function exists:

For all 2-colorings of $[f(n)] \times [f(n)]$ there exists a mono grid of size $n$.

(c) Find a function $f(n, c)$ such that the following is true (and prove it), or show that no such function exists:

For all $c$-colorings of $[f(n, c)] \times [f(n, c)]$ there exists a mono grid of size $n$.

Solution:

(a) False. Consider the coloring $C(a, b) = 1$ if $a < b$, and $C(a, b) = 2$ otherwise.

(b) We give several proofs:

**PROOF ONE:** $f(n) = 2^{2n+1}$

Let COL be a 2-coloring of $[f(n)] \times [f(n)]$.

View COL as a $2^n$-coloring of the rows. There are $2^{2n+1}$ rows so by PHP there are $\frac{2^{2n+1}}{2^n} = 2^{n+1}$ rows that are the same color (we just need $2^n$). Look just at those rows. Assume the color has at least half $R$'s. Each row is $2^n+1$ long, so there is a set of $2^n$ positions where it is $R$. Take the $2^n$ rows, the $2^n$ positions where they are $R$ and you have your mono grid.

**PROOF TWO:** $f(n) = 2^{3n+1}$.

Look at ROW 1: Let $c_1$ be the majority color KILL ALL THOSE WHO DISAGREE. There are now $2^{3n}$ columns.

Look at ROW 2: Let $c_2$ be the majority color KILL ALL THOSE WHO DISAGREE. There are now $2^{3n-1}$ columns.

Keep doing this, until
Look at ROW 2n + 1: Let $c_{2n+1}$ be the majority color KILL. . . . There are now $2^{3n-2n} = 2^n$ columns.

We have $c_1, \ldots, c_{2n+1}$. There are $n$ that are the same color. Take those rows.

**PROOF THREE:** $f(n) = 2^{2n}$ suffices. Let $g(a, b)$ be equal to the number of elements required to find either a mono grid for color 1 of size $a$ or a mono grid for color 2 of size $b$. Then we can construct a mono grid of size $n$ given sufficiently many points as follows: Consider point 1. If there are $g(n-1, n)$ elements $b$ of $B$ such that $COL(1, b) = 1$, let $COL'(1) = 1$ and let $B_1$ be the subset of $N$ such that $COL(1, b) = 1$ for all $b \in B_1$. Otherwise, there will be $g(n, n-1)$ elements $b$ in $B$ such that $COL(1, b) = 2$. In this case $COL'(1) = 2$ and $B_1$ is the set of things such that $COL(1, b) = 2$. Suppose the former case occurred. Then either there is a mono grid $A \times B$ for color 1 of size $(n-1)$ such that $1 \notin A$ and $B \subseteq B_1$, or there is a mono grid $A \times B$ for color 2 of size $n$ such that $1 \notin A$ and $B \subseteq B_1$. Either way, we can then construct a mono grid of size $n$ on the original grid.

A similar argument shows $g(a, b) \leq g(a-1, b) + g(a, b-1)$. We will now show $g(a, b) \leq 2^{a+b}$. First, it is easy to check $g(x, 1) = g(1, x) = x \leq 2^{x+1}$. Now, for $a, b > 1$ we have $g(a, b) \leq g(a-1, b) + g(a, b-1) \leq 2 \cdot 2^{a+b-1} = 2^{a+b}$, as desired. Since $g(n, n) \leq 2^{2n}$, $f(n) = 2^{2n}$ suffices.

(c) $f(n) = c^n$ suffices. Use the previous argument, except instead of $g(a, b) \leq g(a-1, b) + g(a, b-1)$ show that $g_c(a_1, a_2, \ldots, a_c) \leq g_c(a_1-1, a_2, \ldots, a_c) + \cdots + g_c(a_1, a_2, \ldots, a_c-1)$. Check that $g_c(a_1, a_2, a_3, \ldots, a_c) \leq g_{c-1}(a_2, a_3, \ldots, a_c) \leq (c-1)^{(a_2+a_3+\cdots+a_c)} \leq c^{(1+a_2+a_3+\cdots+a_c)}$. Then by induction $g_c(a_1, a_2, \ldots, a_c) \leq c^{(a_1+a_2+\cdots+a_n)}$. Therefore $g_c(n, n, \ldots, n) \leq c^{cn}$, so $f(n) = c^n$ suffices.

**GOTO THE NEXT PAGE**
5. (50 points) In this problem we guide you through a finite version of Mileti’s proof of the infinite can Ramsey Theorem. We work backwards by taking the last part of the proof first.

ADVICE: (1) When the infinite proof asked for an INFINITE subset, here instead take a subset that is of size square root of what we had, (2) make gross overestimates to get this all to work – trying to refine it gets complicated.

**PROBLEM MILLONE**

Find a function $f(n)$ such that the following lemma holds.

**Lemma** Let $COL$ be an $\omega$-coloring of $\left(\binom{f(n)}{2}\right)$. Assume that

- For all $1 \leq i \leq f(n) - 2$, for all $i < k_1 < k_2 \leq f(n)$
  
  \[
  COL(i, k_1) \neq COL(i, k_2).
  \]

- For all $1 \leq i < j \leq f(n) - 1$, for all $k \geq j + 1$,
  
  \[
  COL(i, k) \neq COL(j, k).
  \]

Then there exists a rainbow set of size $n$. (Note that we DO NOT have one yet since $COL(3, 8) = COL(4, 11)$ is possible.)
SOLUTION TO MILLONE

We will pick $f$ later. We define a sequence of $z$'s and a sequence of $H$'s

$z_1 = 1$

$H_1 = \{2, 3, \ldots, f(n)\}$.

Assume that $z_1, \ldots, z_i$ have been chosen and that all of the edges between them are different colors. Let $SETCOL_i$ be the set of colors of edges (there are $i\choose 2$ of them). All of the elements of $H_i$ are $> z_i$. Find the least element $z$ of $H_i$ such that,

$(\forall 1 \leq j \leq i)[COL(z_j, z) \notin SETCOL_i].$

AND

$(\forall 1 \leq j_1 < j_2 \leq i)[COL(z_j_1, z) \neq COL(z_j_2, z)].$

FIRST KEY: The second clause holds for all $z$

SECOND KEY: we need to show that there exists a $z$ satisfying the first clause. We claim that such a $z$ exists within the first $i^3$ elements of $H_i$. Assume, by contradiction, that there is no such $z$. We map each $z \in H_i$ to the REASON it does not work. Map $H_i$ to $\{1, \ldots, i\} \times SETCOL_i$ as follows:

$z \in H_i$. $z$ DID NOT get to be $z_{i+1}$. Hence there is some $j$ (take the least one) such that $COL(z_j, z) = c \in SETCOL_i$. Let $j$ be the least such $j$. Map $z$ to $(j, c)$.

Restrict this map to the first $i^3$ elements of $H_{i-1}$. Now it maps $i^3$ elements to $i \times \binom{i}{3}$ elements, which is $< i^3$. Hence there is $z, z'$ within the first $i^3$ elements of $H_i$ such that there is a $j$ with $COL(z_j, z) = c$ and $COL(z_j, z') = c$. This violated $COL(z_j)$ only has one color coming ot of it.

We now define

$z_{j+1}$ is the $z$ found

$H_{j+1}$ is $H_j$ MINUS all the elements in $H_j$ that were less than $z$ that did not make it. So sad for them :-( .

Since $|H_{i+1}| \geq |H_i| - i^3$ we have
\[ |H_n| \geq |H_0| - 1^2 - 2^3 - 3^3 - \cdots - n^3 \geq |H_0| - n^4. \]

Since we need to do the process \( n \) times take \( f(n) = n^4 \).

**END OF SOLUTION TO MILLONE**

**PROBLEM MILLTWO** Find a function \( g(n) \) such that the following lemma is true: **Lemma** Let \( COL' \) be a coloring of \([g(n)]\) where the colors are of the form \((H, c)\) and \((RB, i)\). Then one of the following must occur:

(a) There exists \( c \) and \( Y \subseteq [g(n)], |Y| \geq n \), such that every element of \( Y \) is colored \((H, c)\).

(b) There exists \( Y \subseteq [g(n)], |Y| \geq n \), such that every element of \( Y \) is colored \((H, *)\) and they all have different second components.

(c) There exists \( i \) and \( Y \subseteq [g(n)], |Y| \geq n \), such that every element of \( Y \) is colored \((RB, i)\).

(d) There exists \( Y \subseteq [g(n)], |Y| \geq n \), such that every element of \( Y \) is colored \((RB, *)\) and they all have different second components.

**SOLUTION TO PROBLEM MILLTWO**

Either \( g(n)/2 \) of the numbers are colored \((H, *)\) or are colored \((RB, *)\). Assume its \((H, *)\) (the other case is similar).

Of these \( g(n)/2 \) elements either there exists \( c \) such that \( \sqrt{g(n)/2} \) are colored \((H, c)\) OR there exists \( \sqrt{g(n)/2} \) with different second components.

So we need \( \sqrt{g(n)/2} \geq n \). We take \( g(n) = 2n^2 \).

**END OF SOLUTION TO PROBLEM MILLTWO**

**GOTO THE NEXT PAGE**
PROBLEM MILLTHREE
Find a function \( h(n) \) such that the following lemma is true: **Lemma**
Let \( COL \) be an \( \omega \)-coloring of \( \left( \left\lfloor h(n) \right\rfloor \right) \) Assume there is a coloring \( COL' \) of \( [h(n)] \) where the colors are of the form \((H, c)\) and \((RB, i)\), and the following holds:

- If \( COL'(x) = (H, c) \) then for all \( z > x \) \( COL(x, z) = c \).
- If \( COL'(x) = (RB, i) \) then for all \( z_1 \neq z_2 > x \) \( COL(x, z_1) \neq COL(x, z_2) \).
- If \( COL'(x) = (RB, i) \) and \( COL'(y) = (RB, i) \) then for all \( z > \max\{x, y\} \) \( COL(x, z) = COL(y, z) \).
- If \( COL'(x) = (RB, i) \) and \( COL'(y) = (RB, j) \) (with \( i \neq j \)) then for all \( z > \max\{x, y\} \) \( COL(x, z) \neq COL(y, z) \).

Then one of the followings holds:
(a) There is a homog set of size \( n \).
(b) There is a min-homog set of size \( n \).
(c) There is a max-homog set of size \( n \).
(d) There is a rainbow set of size \( n \).

**SOLUTION TO MILLTHREE**
By the solution to MILLTWO one of the following holds:

(a) There are \( \sqrt{\frac{h(n)}{2}} \) with \((H, c)\). Then there is a homog set of size \( \sqrt{\frac{h(n)}{2}} \) so we need \( h(n) \geq 2n^2 \).
(b) There are \( \sqrt{\frac{h(n)}{2}} \) with \((H, *)\), all different second parts. Then there is a min-homog set of size \( \sqrt{\frac{h(n)}{2}} \) so we need \( h(n) \geq 2n^2 \).
(c) There are \( \sqrt{\frac{h(n)}{2}} \) with \((RB, i)\) Then there is a max-homog set of size \( \sqrt{\frac{h(n)}{2}} \) so we need \( h(n) \geq 2n^2 \).
(d) There are \( \sqrt{\frac{h(n)}{2}} \) with \((RB, *)\), all different second parts. You DO NOT have a rainbow set! You use the solution to MILLONE to get a rainbow set of size \( (\sqrt{\frac{h(n)}{2}})^{1/4} = (\frac{h(n)}{2})^{1/8} \).

So take \( h(n) = 2n^8 \).
PROBLEM MILLFOUR

Find a function $BILL(n)$ (sorry, I’m running out of letters) such that the following lemma is true: **Lemma:** Let $COL$ be a $\omega$-coloring of $\binom{[BILL(n)]}{2}$ Then there is a subset of $[BILL(n)]$ of size $n$ and a coloring $COL'$ of that subset, where the colors are of the form $(H, c)$ and $(RB, i)$, such that the following holds:

- If $COL'(x) = (H, c)$, then for all $z > x$, $COL(x, z) = c$.
- If $COL'(x) = (RB, i)$, then for all $z_1, z_2 > x$, $COL(x, z_1) \neq COL(x, z_2)$.
- If $COL'(x) = (RB, i)$ and $COL'(y) = (RB, i)$, then for all $z > \max\{x, y\}$, $COL(x, z) = COL(y, z)$.
- If $COL'(x) = (RB, i)$ and $COL'(y) = (RB, j)$ (with $i \neq j$), then for all $z > \max\{x, y\}$, $COL(x, z) \neq COL(y, z)$.
SOLUTION TO MILLFOUR

\[ V_0 = [BILL(n)] \]
\[ x_1 = 1 \]

If \( (\exists c)\{|v \in V_0 \mid COL(x_1, v) = c\}| \geq \sqrt{|BILL(n)|} \) then:

- \( c_1 = (H, c) \)
- \( V_1 = \{v \in V_0 \mid COL(x_1, v) = c\} \). (Note that \(|V_1| \geq \sqrt{|BILL(n)|}\))

If \( (\forall c)\{|v \in V_0 \mid COL(x_1, v) = c\}| < \sqrt{|BILL(n)|} \) then:

- \( V_1 = \{v \in V_0 \mid (\exists c)[COL(x_1, v) = c \land (\forall x_1 < u < v)[COL(x_1, u) \neq c]]\} \) (so \( v \) is the first first with \( COL(x_1, v) = c \). Hence there will only be ONE \( v \) with \( COL(x_1, v) = c \).) (Note that \(|V_1| \geq \sqrt{|BILL(n)|}\))
- \( c_1 = (RB, 1) \). (The 1 marks that this is the first rainbow-color assigned.)

Let \( i \geq 2 \), and assume that \( V_{i-1} \) is defined. We define \( x_i, c_i, \) and \( V_i \):

- \( x_i \) gets the least element of \( V_{i-1} \).
- For all colors \( c \) let \( Y_c = \{x \in V_{i-1} : COL(x_i, x) = c\} \)
- Also let: \( Y_\omega = \{x \in V_{i-1} : (\forall y \in V_{i-1}, y < x)[COL(x_i, x) \neq COL(x_i, y)]\} \)
  (So all colors coming out of \( x \) are different.
- If there exists \( c \) such that \(|Y_c| \geq \sqrt{|V_{i-1}|}\) then
  \[ c_i = (H, c) \]
  \[ V_i = Y_c \]

If no such \( c \) exist then there exists \( Y_\omega \) with \(|Y_\omega| \geq \sqrt{|V_{i-1}|}\). with all of the vertices coming out of it being different colors. We initially take \( V_i = Y_\omega \)

But we may thin it out. And we haven’t colored \( x_i \) yet.
Do the following:
For all $1 \leq j \leq i - 1$ such that $COL'(x_j) = (RB, k)$ for some $k$ then:

(a) If $|\{y \in Y : COL(x_j, y) = COL(x_i, y)\}| \geq \sqrt{|V_i|}$ then let $V_i$ be this set and let $c_i = c_j$. (So $COL'(x_i)$ will be of the form $(RB, k)$ for some $k$). You are done and do not go to the next $j$.

(b) If $|\{y \in Y : COL(x_j, y) = COL(x_i, y)\}| < \sqrt{|V_i|}$ then let $V_i$ be the $Y$ minus those vertices.

If Case 1 ever happens then we are done. If Case 2 always happens then note that $x_i$ disagrees with every $x_j$ on every element $> x_i$. We $c_i$ with $(RB, k)$ where $k$ is the least number not used for a rainbow color yet.

**END OF CONSTRUCTION**

The KEY for us is how big is $V_i$.

In the worst case we keep on subtracting $\sqrt{|V_i|}$ vertices and then at the very last stage take a square root. Even though $|V_i|$ keeps getting smaller within a stage we won’t use this (so our results are not as good as they could be).

Lets start at the beginning
We have $V_{i-1}$.

We do the $Y_\omega$ thing
We now have a set of size $\sqrt{|V_{i-1}|}$.

We then subtract $|V_{i-1}|^{1/4}$ $i$ times.

So we have

$$|V_i| \geq \sqrt{|V_{i-1}|^{1/2} - i|V_{i-1}|^{1/4}}$$

To simplify we will assume $i|V_i|^{1/4} \leq \frac{|V_i|^{1/2}}{2}$. (we later make sure that all $|V_i| \geq 16i^4$ so this true). Hence

$$|V_i| \geq \sqrt{|V_{i-1}|^{1/2} - i|V_{i-1}|^{1/4}} \geq \sqrt{\frac{|V_{i-1}|^{1/2}}{2}}$$

So we have
\[ |V_i| \geq \frac{|V_{i-1}|^{1/4}}{\sqrt{2}} \]

We get really lazy here and make this even easier to deal with by assuming
\[ |V_i| \geq \frac{|V_{i-1}|^{1/4}}{\sqrt{2}} \geq |V_{i-1}|^{1/5} \] (we later make sure that all \( |V_i| \geq 4^5 \) to make this true).

\[ |V_i| \geq |V_{i-1}|^{1/5} \]

\[ |V_n| \geq |V_0|^{(1/5)^n} \]

So we need \( |V_0|^{(1/5)^n} \geq 4^5 \), so \( |V_0| \geq 4^{5^n+1} \).

We also need \( |V_0|^{(1/5)^n} \geq 16n^4 \), so \( |V_0| \geq (16n^4)^{5^n} \)

so we can take \( BILL(n) = 4^{5^n+1} + (16n^4)^{5^n} \)

**END OF SOLUTION TO MILLFOUR**

**GOTO THE NEXT PAGE**
**PROBLEM MILLFIVE** Put all of this together to (easily) find a function \( CR(n) \) (for Can Ramsey) such that the following theorem is true:

**Theorem** Let \( COL \) be an \( \omega \)-coloring of \( \binom{|CR(n)|}{2} \). Then one of the following holds:

(a) There is a homog set of size \( n \).
(b) There is a min-homog set of size \( n \).
(c) There is a max-homog set of size \( n \).
(d) There is a rainbow set of size \( n \).

**SOLUTION TO MILLFIVE**

Let \( CR(n) = BILL(2n^8) = 4^{5n^8+1} + (256n^{32})^{5n^8} \).

We leave it to the reader to see that this works.

**END OF SOLUTION TO MILLFIVE**
6. (25 points) (This is a NEW problem – nothing to do with Finite Can Ramsey.) Let \((L, \preceq)\) be a well quasi order. Let \(2^{\text{fin}L}\) be the set of FINITE subsets of \(L\). We DEFINE an order \(\preceq'\) on \(2^{\text{fin}L}\):

\[ A \preceq' B \text{ if there is an injection } f \text{ from } A \text{ to } B \text{ such that } x \preceq f(x). \]

(\(\emptyset \preceq' B\) is always true: use the empty function and the condition holds vacuously.)

Show that \((2^{\text{fin}L}, \preceq')\) is a well quasi order.

(NOTE- this proof will use that wqo are closed under cross product, but the proof I have does not use Ramsey Theory directly.)

SOLUTION TO PROBLEM SIX

Throughout ‘smallest’ means smallest CARDINALITY of a set.

Assume, BWOC, that \((2^{\text{fin}L}, \preceq')\) is a NOT a wqo.

Let \(A_1\) be the smallest set that begins a bad sequence.

Let \(A_2\) be the smallest set that is the second element of a bad sequence that begins with \(A_1\)

For all \(i \geq 3\)

Let \(A_i\) be the smallest set that is the \(i\)th element of a bad sequence that begins with \(A_1, A_2, \ldots, A_{i-1}\).

Note that

\[ A_1, A_2, A_3, \ldots \]

is a minimal bad sequence.

None of the \(A_i\)’s can be empty since its a bad sequence.

Let \(B_i\) be \(A_i\) minus an element.

The elements are picked arb, however lets call the set of such elements MINUS.

Let \(B = \{B_1, B_2, \ldots\}\).

Claim: \(B\) with the order \(\preceq'\) is a wqo

Proof of Claim: Assume, BWOC, that there is a bad sequence:

\[ B_{i_1}, B_{i_2}, \ldots \]
We can assume that $i_1$ is the smallest index that appears (take the smallest one that appears and start there). Aside from that we DO NOT know anything about the order of the $i_j$’s.

Look at the sequence

$$A_1, A_2, \ldots, A_{i_1-1}, B_{i_1}, B_{i_2}, \ldots$$

(NO TED we DO NOT KNOW, NOR DO WE THINK that $i_1 < i_2 < \cdots$)

We show this is a BAD sequence.

(a) Since $A_1, A_2, \ldots$ is a bad sequence there will be no uptick in the first $i_1 - 1$ elements of the sequence.

(b) Since $B_{i_1}, B_{i_2}, \ldots$ is a bad sequence there will be on uptick in the elements after $A_{i_1-1}$.

(c) Assume, BWOC, that we have $i < i_j$ and $A_i \preceq' B_{i_j}$. Take the injection from $A_i$ to $B_{i_j}$ and view it as an injection from $A_i$ to $A_{i_j}$. Hence $i < i_j$ and $A_i \preceq A_{i_j}$. Hence we have an uptick in the BAD SEQUENCE $A_1, A_2, \ldots$. This is a contradiction.

SO

$$A_1, A_2, \ldots, A_{i_1-1}, B_{i_1}, B_{i_2}, \ldots$$

is a bad sequence. Look at its $i_1$ element. Recall how $A_{i_1}$ was defined:

Let $A_{i_1}$ be the smallest set that is the $i_1$th element of a bad sequence that begins with $A_1, A_2, \ldots, A_{i_1-1}$.

BUT we are now looking at a bad sequence that begins with

$$A_1, A_2, \ldots, A_{i_1-1}$$

with $i_1$th element $B_{i_1}$, and $|B_{i_1}|$ is $A_{i_1}$ with one element missing so it is SMALLER. This is a contradiction.

So $B$ with $\preceq'$ is a wqo.

**End of Proof of Claim**

SO $B$ under $\preceq'$ is a wqo

$MINUS$ under $\preceq$ is a subset of a wqo so its a wqo.
So $\mathcal{B} \times MINUS$ is a wqo.

Look at the original bad sequence (Sounds like a rap singer’s nickname the original badass sequence! - Maybe he or she could do a rap song about badass sequences – it could not be worse than the BW “rap”).

SEQONE: $A_1, A_2, \ldots$,

View this as

SEQTWO: $(B_1, b_1), (B_2, b_2), \ldots$

Where $A_i = B_i \cap \{b_i\}$.

Since SEQTWO has an uptick, SEQONE has an uptick.