An Application of Ramsey Theory to in Multiparty Communication Complexity
Exposition By William Gasarch

1 Introduction

Multiparty communication complexity was first defined by Chandra, Furst, and Lipton [6] and used to obtain lower bounds on branching programs. Since then it has been used to get additional lower bounds and tradeoffs for branching programs [1, 3], lower bounds on problems in data structures [3], time-space tradeoffs for restricted Turing machines [1], and unconditional pseudorandom generators for logspace [1].

All results in this paper are from [6] or can be easily derived from their techniques unless otherwise specified.

Def 1.1 Let \( f : \{(0,1)^n\}^k \rightarrow X \). Assume, for \( 1 \leq i \leq k \), \( P_i \) has all of the inputs except \( x_i \).

Let \( d(f) \) be the total number of bits broadcast in the optimal deterministic protocol for \( f \). At the end of the protocol all parties must know the answer. This is called the multiparty communication complexity of \( f \). The scenario is called the forehead model.

Note 1.2 Note that there is always the \( n+1 \)-bit protocol of (1) \( P_1 \) broadcasts \( x_2 \), (2) \( P_2 \) computes and broadcasts \( f(x_1, \ldots, x_k) \). The cases of interest are when \( d(f) \ll n \).

We will need the following lemmas about multiparty protocols. The first one is the \( k = 3 \) case of the second one. We leave it for an exercise.

Lemma 1.3 Let \( P \) be a multiparty protocol for a function \( f : \{(0,1)^n\}^k \times \{(0,1)^n\} \times \{(0,1)^n\} \rightarrow X \).

1. Let \( TRAN \) be a possible transcript of the protocol \( P \). There exists \( A_1, A_2, A_3 \subseteq \{(0,1)^n\} \) such that, for all \( x_1, x_2, x_3 \in \{(0,1)^n\} \) the following holds: The protocol \( P \) on input \( (x_1, x_2, x_3) \) produces transcript \( TRAN \) iff \( (x_1, x_2, x_3) \in A_1 \times A_2 \times A_3 \).

2. Let \( x_1, x_2, x_3 \in \{(0,1)^n\}, \sigma_1, \sigma_2, \sigma_3 \in \{(0,1)^n\}^3 \), \( TRAN \) be a transcript. Assume that \( \sigma_1 \) has \( x_1 \) as its first element, \( \sigma_2 \) has \( x_2 \) as its second element, \( \sigma_3 \) has \( x_3 \) as its third element. (In symbols, if * means we don’t care about the element, then

\[
\begin{align*}
\sigma_1 &= (x_1, *, *) \\
\sigma_2 &= (*, x_2, *) \\
\sigma_3 &= (*, *, x_3).
\end{align*}
\]

Further assume that \( \sigma_1, \sigma_2, \sigma_3 \) all produces transcript \( TRAN \). Then \( (x_1, x_2, x_3) \) produces transcript \( TRAN \).

Lemma 1.4 Let \( P \) be a multiparty protocol for a function \( f : \{(0,1)^n\}^k \rightarrow X \).
1. Let $TRAN$ be a possible transcript of the protocol $P$. There exists $A_1, \ldots, A_k \subseteq \{0, 1\}^n$ such that, for all $x_1, \ldots, x_k \in \{0, 1\}^n$ the following holds: The protocol $P$ on input $(x_1, \ldots, x_k)$ produces transcript $TRAN$ iff $(x_1, \ldots, x_k) \in A_1 \times \cdots \times A_k$.

2. Let $x_1, \ldots, x_k \in \{0, 1\}^n$, $\sigma_1, \ldots, \sigma_k \in \{(0, 1)^n\}^k$, $TRAN$ be a transcript. Assume that $\sigma_i$ has $x_i$ as its $i$th element. Further assume that each $\sigma_i$ produces transcript $TRAN$. Then $(x_1, \ldots, x_k)$ produces transcript $TRAN$.

We will study the following function.

**Def 1.5** Let $n \in \mathbb{N}$. We define $EQ_{2^n} : \{0, 1\}^n \times \{0, 1\}^n \times \{0, 1\}^n$ as follows (interpreting the three inputs as numbers in binary):

$$EQ_{2^n}(x, y, z) = \begin{cases} YES & \text{if } x + y + z = 2^n \\ NO & \text{if } x + y + z \neq 2^n \end{cases}$$ (1)

We will first establish a connection between $d(EQ_{2^n})$ and some concepts in Ramsey Theory. We will then use results from Ramsey Theory to obtain upper and lower bounds on $d(EQ_{2^n})$. The lower bounds will be applied to obtain lower bounds on branching programs. Here is what we will show.

1. $d(EQ_{2^n}) \leq \sqrt{\log(2^n)} = \sqrt{n}$ (First proven by Chandra, Furst, Lipton [6].) (This is somewhat surprising since it would seem the best you could do is have Alice yell to Bob what her bits are.)

2. $d(EQ_{2^n}) \geq \omega(1)$ (First proven by Chandra, Furst, Lipton [6].)

3. $d(EQ_{2^n}) \geq \log \log 2^n + \Omega(1) = \log \log n + \Omega(1)$ (First proven by Beigel, Gasarch, Glenn [5].)

## 2 Connections Between Multiparty Comm. Comp. and Ramsey Theory

In this section we review the connections between the multiparty communication complexity of $f$ and Ramsey Theory that was first established in [6].

**Def 2.1** Let $c, T \in \mathbb{N}$.

1. A proper $c$-coloring of $[T] \times [T]$ is a function $COL : [T] \times [T] \rightarrow [c]$ such that there do not exist $x, y \in [T]$ and $\lambda \in [T - 1]$ such that

$$COL(x, y) = COL(x + \lambda, y) = COL(x, y + \lambda)$$

Another way to look at this: In a proper coloring there cannot be three vertices that (a) are the same color, and (b) are the corners of a right isosceles triangle with legs parallel to the axes and hypotenuse parallel to the line $y = -x.$)
2. Let \( \chi(T) \) be the least \( c \) such that there is a proper \( c \)-coloring of \([T] \times [T]\).

**Theorem 2.2** Let \( 2^n : \mathbb{N} \rightarrow \mathbb{N} \).

1. \( d(\text{EQ}_n^n) \leq 2 \log(\chi(2^n)) + O(1) \).
2. \( d(\text{EQ}_n^n) \geq \log(\chi(2^n)) + \Omega(1) \).

**Proof:**

1) Let \( \text{COL} \) be a proper \( c \)-coloring of \([2^n] \times [2^n]\). We represent elements of \([c]\) by \( \log(\chi(2^n)) + O(1) \) bit strings. \( P_1, P_2, P_3 \) will all know \( \text{COL} \) ahead of time. We present a protocol for this problem for which the communication is \( 2 \log(\chi(2^n)) + O(1) \). We will then show that it is correct.

1. \( P_1 \) has \( y, z \). \( P_2 \) has \( x, z \). \( P_3 \) has \( x, y \).
2. \( P_1 \) calculates \( x' \) such that \( x' + y + z = 2^n \). (If no such \( x' \) exists then output NO and thats the end of the protocol.) \( P_1 \) broadcasts \( \sigma_1 = \text{COL} (x', y) \).
3. \( P_2 \) calculates \( y' \) such that \( x + y' + z = 2^n \). (If no such \( y' \) exists then output NO and thats the end of the protocol.) \( P_2 \) broadcasts \( \sigma_2 = \text{COL} (x, y') \).
4. \( P_3 \) looks up \( \sigma_3 = \text{COL} (x, y) \). \( P_3 \) broadcasts YES if \( \sigma_1 = \sigma_2 = \sigma_3 \) and NO otherwise. (We will prove later that these answers are correct.)

**Claim 1:** If \( \text{EQ}_n^n(x, y, z) = \text{YES} \) then \( P_1, P_2, P_3 \) will all think \( \text{EQ}_n^n(x, y, z) = \text{YES} \).

**Proof:** If \( \text{EQ}_n^n(x, y, z) = \text{YES} \) then \( x'_1 = x_1, x'_2 = x_2, \) and \( x'_3 = x_3 \). Hence \( \sigma_1 = \sigma_2 = \sigma_3 \) Therefore \( P_1, P_2, P_3 \) all think \( \text{EQ}_n^n(x, y, z) = \text{YES} \).

End of proof of Claim 1.

**Claim 2:** If \( P_1, P_2, P_3 \) all think that \( \text{EQ}_n^n(x, y, z) = \text{YES} \) then \( \text{EQ}_n^n(x, y, z) = \text{YES} \).

**Proof:** Assume that \( P_1, P_2, P_3 \) all think \( \text{EQ}_n^n(x, y, z) = \text{YES} \).

Hence

\[ \text{COL} (x_1, x_2) = \text{COL} (x'_1, x_2) = \text{COL} (x_1, x'_2). \]

We call this the **The Coloring Equation**.

Assume

\[ x_1 + x_2 + x_3 = \lambda. \]

We show that \( \lambda = 2^n \).

By the definition of \( x'_1 \)
\[ x'_1 + x_2 + x_3 = 2^n. \]

Hence

\[ x'_1 + (x_1 + x_2 + x_3) - x_1 = 2^n. \]

\[ x'_1 + \lambda - x_1 = 2^n. \]

\[ x'_1 - x_1 = 2^n - \lambda \]

\[ x'_1 = x_1 + 2^n - \lambda \]

By the same reasoning

\[ x'_2 = x_2 + 2^n - \lambda. \]

Hence we can rewrite The Coloring Equation as

\[ \text{COL} (x_1, x_2) = \text{COL} (x_1 + 2^n - \lambda, x_2) = \text{COL} (x_1, x_2 + 2^n - \lambda). \]

Since \( \text{COL} \) is a proper coloring, \( 2^n - \lambda = 0 \), so \( \lambda = 2^n \).

End of proof of Claim 2.

2) Let \( P \) be a protocol for \( \text{EQ}_{2^n}^2 \). Let \( d \) be the maximum number of bits communicated. Note that the number of transcripts is bounded by \( 2^d \). We use this protocol to create a proper \( 2^d \)-coloring of \( [2^n] \times [2^n] \).

We define \( \text{COL} (x, y) \) as follows. First find \( z \) such that \( x + y + z = 2^n \). Then run the protocol on \( (x, y, z) \). The color is the transcript produced.

Claim 3: \( \text{COL} \) is a proper coloring of \( [2^n] \times [2^n] \).

Proof: Let \( \lambda \in [2^n] \) be such that

\[ \text{COL} (x, y) = \text{COL} (x + \lambda, y) = \text{COL} (x, y + \lambda). \]

We denote this value \( \text{TRAN} \) (for Transcript). We show that \( \lambda = 0 \).

Let \( z \) be such that

\[ x + y + z = 2^n. \]

Since

\[ \text{COL} (x, y) = \text{COL} (x + \lambda, y) = \text{COL} (x, y + \lambda). \]

We know that the following tuples produce the same transcript \( \text{TRAN} \):
• \((x, y, z)\).

• \((x + \lambda, y, z - \lambda)\).

• \((x, y + \lambda, z - \lambda)\).

All of these input produce the same transcript \(TRAN\) and this transcript ends with a YES. By Lemma 1.3.2 the tuple \((x, y, z - \lambda)\) also goes to \(TRAN\). Hence \(x + y + z - \lambda = 2^n\). Since \(x + y + z = 2^n\) we have \(\lambda = 0\).

*End of Proof of Claim 3*

We now have a really odd situation. We have \(d(EQ\, 2^n) = \Theta(lg(\chi(2^n)))\)

**YEAH:** We have upper and lower bounds that match up to a multiplicative constant!

**BOO:** We don’t know that the function IS.

In the next two sections we get upper bounds and lower bounds on \(lg(\chi(2^n))\).

3 Upper Bounds

We need to properly color \([2^n] \times [2^n]\) and keep the number of colors down.

4 Lower Bounds

4.1 An \(\omega(1)\) Lower Bound for \(d(EQ_n^{2^n})\)

We will need the following theorem from Ramsey Theory.

**Theorem 4.1** For all \(c\) there exists \(T\) such that, there are no proper \(c\)-colorings of \([T] \times [T]\).

Theorem 4.1 can be proven several ways. We enumerate them:

1. This can be proven from van der Waerden’s theorem.

2. This can be proven by the same techniques as van der Waerden’s theorem.

3. This follows from the Galai-Witt Theorem. This generalizes to coloring \([T]^k\).

4. We will give a concrete lower bound (rather than \(\omega(1)\)) and is in Section 4.2. Other ways generalize to \(k\) variables.

**Theorem 4.2** If \(\lim_{n \to \infty} 2^n = \infty\) then \(d(EQ_n^{2^n}) = \omega(1)\).
Proof: By Theorem 2.2
\[ d(\text{EQ}_{2^n}) \geq \lg(\chi(2^n)) + \Omega(1). \]
Hence we need to show that \( \chi(T) \) is not bounded by a constant (as \( T \) goes to infinity).
Assume, by way of contradiction, that there exists \( c \) such that, for all \( T \), there is a proper \( c \)-coloring of \([T] \times [T] \). This contradicts Theorem 4.1. 

We will need to look at \( k \)-party protocols for the following function.
\[
\text{MOD}_{n,k}^{2^n} : (\{0, 1\}^n)^k \rightarrow \{0, 1\}
\]
\[ \text{MOD}_{n,k}^{2^n}(x_1, \ldots, x_k) = \begin{cases} 
1 & \text{if } \sum_{i=1}^{k} x_i = 2^n \\
0 & \text{otherwise.} \end{cases} \tag{2} \]

The following can be proven in a manner similar to the \( k = 3 \) case.

**Theorem 4.3** Fix \( k \). If \( \lim_{n \to \infty} 2^n = \infty \) then \( d(\text{MOD}_{n,k}^{2^n}) = \omega(1) \).

### 4.2 An \( \Omega(\log \log \log 2^n) \) Lower Bound for \( d(\text{EQ}_{n}^{2^n}) \)

The following combinatorial lemma will allow us to prove a lower bound on \( d(\text{EQ}_{n}^{2^n}) \). This lemma is a reworking of a theorem of Graham and Solymosi [9].

**Lemma 4.4**

1. \( \chi(2^n) \geq \Omega(\log \log 2^n) \).
2. \( d(\text{EQ}_{n}^{2^n}) \geq \log \log \log 2^n + \Omega(1). \) (This follows from part 1 and Theorem 2.2.)

**Proof:** Assume that COL is a proper \( c \)-coloring of \([2^n] \times [2^n] \). We find sets \( X_1, Y_1 \subseteq [2^n] \times [2^n] \) such that COL restricted to \( X_1 \times Y_1 \) uses \( c - 1 \) colors. We will iterate this process to obtain \( X_c, Y_c \) such that COL restricted to \( X_c \times Y_c \) uses 0 colors. Hence \( |X_c| = 0 \) which will yield \( c = \Omega(\log \log 2^n) = \Omega(\log \log n) \).

For \( 0 \leq s \leq c \) we define \( X_s, Y_s, h_s, \text{USED-COL}_s \).

1. \( X_0 = Y_0 = [2^n] \). \( h_0 = |X_0| = |Y_0| = 2^n \). \( \text{USED-COL}_0 = [c] \).

2. Assume \( X_s, Y_s, h_s \) are defined and inductively \( \text{USED-COL}_s = [c - s] \) (we will be renumbering to achieve this). Also assume that Partition \( X_s \times Y_s \) (which is of size \( h_s^2 \)) into sets \( P_a \) indexed by \( a \in [2^n] \) defined by

\[
P_a = \{(x, y) \in X_s \times Y_s \mid x + y = a\}. \]

(\( P_a \) is the \( a \)th anti-diagonal.) There exists an \( a \) such that \( |P_a| \geq [h_s^2/2^n] \). There exists a color, which we will take to be \( c - s \) by renumbering, such that at least \( [[h_s^2/2^n]]/c \) of the elements of \( P_a \) are colored \( c - s \). (We could use \( c - s \) in the denominator but we do not need to.) Let \( m = [[h_s^2/2^n]]/c \). Let \( \{(x_1, y_1), \ldots, (x_m, y_m)\} \) be \( m \) elements of \( P_a \) such that, for \( 1 \leq i \leq m \), COL \( (x_i, y_i) = c - s \). We will later show that, for all \( i \neq j \), COL \( (x_i, y_j) \neq c - s \).
3. Let

\[ h_{s+1} = m' = \lceil m/3 \rceil \]
\[ X_{s+1} = \{x_1, x_2, \ldots, x_{m'}\} \]
\[ Y_{s+1} = \{y_{m+1-m'}, \ldots, y_m\} \]
\[ \text{USED-COL}_{s+1} = \lfloor c - (s + 1) \rfloor \]

Note that for all \((x_i, y_j) \in X_{s+1} \times y_j \in Y_{s+1}, i < j\) hence \(i \neq j\). Since we will show that for all \(i \neq j, \ \text{COL} (x_i, y_j) \neq c - s\), we will have that, for all \((x, y) \in X_{s+1} \times y_j \in Y_{s+1}, \ \text{COL} (x, y) \neq c - s\).

**Claim 1:** For all \(i \neq j, x_i \neq x_j\) and \(y_i \neq y_j\).

**Proof:** If \(x_i = x_j\) then

\[ x_j + y_j = a = x_i + y_i = x_j + y_i. \]

Hence \(y_j = y_i\). Therefore \((x_i, y_i) = (x_j, y_j)\). This contradicts \(P_a\) having \(m\) distinct points.

The proof that \(y_i \neq y_j\) is similar.

*End of Proof of Claim 1*

**Claim 2:** For all \(i \neq j, \ \text{COL} (x_i, y_j) \neq c - s\).

**Proof:** Assume, by way of contradiction, that \(\text{COL} (x_i, y_j) = c - s\). Note that

\[ \text{COL} (x_i, y_j) = \text{COL} (x_i, y_i) = \text{COL} (x_j, y_j) = c - s. \]

We want a \(\lambda \neq 0\) such that \(y_i = y_j + \lambda\) and \(x_j = x_i + \lambda\). Using that \(x_i + y_i = x_j + y_j = a\) we can take \(\lambda = (x_j + y_i - a)\). The element \(\lambda \neq 0\): if \(\lambda = 0\) then one can show \(y_i = y_j\), which contradicts Claim 1.

We now have

\[ \text{COL} (x_i, y_j) = \text{COL} (x_i + \lambda, y_j) = \text{COL} (x_i, y_j + \lambda). \]

This violates \(\text{COL}\) being a proper coloring.

*End of Proof of Claim 2*

Note that, by Claim 2 above

\[ \{ \text{COL} (x, y) \mid x \in X_{s+1}, y \in Y_{s+1} \} \subseteq \text{USED-COL}_{s+1}. \]

Look at what happens at stage \(c\). \(|X_c| = |Y_c| = h_c\) and \(\text{COL}\) restricted to \(X_c \times Y_c\) uses 0 colors. The only way this is possible is if \(h_c = 0\). We will see that this implies \(c = \Omega(\log \log 2^n)\).

We have \(h_0 = 2^n\) and

\[ h_{s+1} = \left\lceil \left\lceil \frac{h_s^2}{2^n} \right\rceil / c \right\rceil / 3 \geq \frac{h_s^2}{3c2^n}. \]
We show that for $s \in \mathbb{N}$, $h_s \geq \frac{2^n}{(3c)^{2s-1}}$.

Claim 3: $(\forall s)[h_s \geq \frac{2^n}{(3c)^{2s-1}}]$.

**Base Case:** $h_0 = 2^n \geq \frac{2^n}{(3c)^0} = 2^n$.

**Induction Step:** Assume $h_s \geq \frac{2^n}{(3c)^{2s-1}}$. Since $h_{s+1} \geq (h_s)^2/3c2^n$ we have, by the induction hypothesis

$$h_{s+1} \geq \frac{(h_s)^2}{3c2^n} \geq \frac{(2^n)^2}{(3c)^{2s+1}-1}.$$

**End of proof of Claim 3**

Taking $s = c$ we obtain $h_c \geq \frac{2^n}{(3c)^{2c-1}}$. Hence there is a set of $h_c^2$ points that are 0-colored. Therefore $h_c < 1$. This yields $c = \Omega(\log \log 2^n)$.

**References**


