An Exposition of Ramsey's Result in Logic
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1 Introduction

In Ramsey's celebrated paper [5] (see also, [2],[3],[4]) his goal was to solve a problem in logic. In this note we discuss what he proved in logic.

We will first state and prove his theorem in logic for undirected graphs (no self loops), and then we will state and prove his theorem in logic for colored hypergraphs.

Def 1.1

- A *graph* is a pair (V, E) where E is a subset of unordered pairs of distinct elements of V. V is referred to as the set of *vertices*. E is referred to as the set of *edges*.
- A *clique* in a graph is a set of vertices such that every pair of vertices in it has an edge.
- An *independent set* in a graph is a set of vertices such that every pair of vertices in it does not have an edge.

The following is a subcase of Ramsey's Combinatorial theorem.

Theorem 1.2 *For all m there exists a number $R(m)$ such that, for every graph on $R(m)$ vertices, there is either a clique or independent set of size m.*

Note 1.3 It is well known that $2^{m/2} \le R(m) \le 2^{2m}$. A more sophisticated proof, by David Conlon [1] yields, for all k, $n \ge k^{-D\frac{\log k}{\log\log k}}\binom{2k}{k}$ suffices, where D is some constant. A simple probabilistic argument shows that $n \ge (1+o(1))\frac{1}{e\sqrt{2}})k2^{k/2}$ is necessary. A more sophisticated argument shown by Spencer [6] (see [3]) shows $n \ge (1+o(1))\frac{\sqrt{2}}{e}k2^{k/2}$ is necessary.

Def 1.4 A sentence is in *the language of graphs* if it only has the usual logical symbols, E a 2-ary predicate, and $=$. We will interpret such sentences as being about undirected graphs with no self loops. Hence we will implicitly assume (1) $E(x, y)$ iff $E(y, x)$ and (2) $\neg E(x, x)$.

Def 1.5 If ϕ is a sentence in the language of graphs then spec(ϕ) is the set of all n such that there is an undirected graph with no self-loops on n vertices where ϕ is true.

Convention 1.6 For ease of notation we make the following conventions.

- If there is a contiguous string of the same type of quantifiers then all of the variables in it are distinct. Hence
$$(\exists x_1)(\exists x_2)(\forall y_1)(\forall y_2)[\phi(x_1, x_2, y_1)]$$
actually means
$$(\exists x_1)(\exists x_2 \ne x_1)(\forall y_1)(\forall y_2 \ne y_1)[\phi(x_1, x_2, y_1)]$$

- There are no self-loops. Hence $E(x,y)$ means $E(x,y) \wedge x \neq y$.
- E is symmetric. So $E(x,y)$ means $E(x,y) \wedge E(y,x)$.

Example 1.7

1. $$\phi = (\forall x)(\forall y)[E(x,y)].$$

   This states that every pair of distinct vertices has an edge. For all n this is satisfied by K_n. Hence, $\mathrm{spec}(\phi) = \mathsf{N}$.

2. $$\phi = (\exists x,y,z)(\forall w)[E(w,x) \wedge E(w,y) \wedge E(w,z)].$$

   ϕ states that there are three distinct vertices x,y,z such that every $w \notin \{x,y,z\}$ is connected to x and y and z. For all $n \geq 0$ $K_{n,3}$ satisfied ϕ. No graph on 0,1, or 2 vertices satisfies ϕ. Hence, $\mathrm{spec}(\phi) = \{3,4,5,\ldots,\}$. (Note that $K_{0,3}$ satisfies ϕ vacuously.)

3. $$\phi = (\exists x_1)(\exists x_2)(\forall y)[x_1 = y \vee x_2 = y].$$

   ϕ is satisfied by all graphs on 2 vertices; however, it is not satisfied by any other graphs. Hence $spec(\phi) = \{2\}$.

Note that in all three examples $\mathrm{spec}(\phi)$ was either co-finite or finite. We will later see that, for all ϕ, this is the case.

2 Definitions and a Lemma Needed for the Graph case

Lemma 2.1

1. *The following is decidable: Given a sentence ϕ and a graph G, determine if ϕ is true in G.*
2. *The following is decidable: Given a sentence ϕ and a number n, determine if $n \in \mathrm{spec}(\phi)$.*

Proof: Use brute force. ▮

We will use Lemma 2.1 without comment.

3 Ramsey's Theorem in Logic on Graphs

The following is a simple case of what Ramsey proved.

Theorem 3.1 *The following function is computable: Given ϕ, a sentence in the language of graphs of the form*

$$(\exists x_1)\cdots(\exists x_n)(\forall y_1)\cdots(\forall y_m)[\psi(x_1,\ldots,x_n,y_1,\ldots,y_m)]$$

output $\mathrm{spec}(\phi)$*. (*$\mathrm{spec}(\phi)$ *will be a finite or cofinite set; hence it will have an easy description.)*

Proof:
Claim 1: If G satisfies ϕ and $x_1, \ldots, x_n$ are the witnesses then any induced subgraph H of G that contains $x_1, \ldots, x_n$ satisfies ϕ.
Proof of Claim 1:

The statement

$$(\forall y_1) \cdots (\forall y_m)[\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)]$$

is true in H since it is true in G and now there are just less cases to check.
End of Proof of Claim 1

Claim 2:

1. If there exists $N_0 \geq n + 2^n R(m)$ such that $N_0 \in \text{spec}(\phi)$ then
$$\{n+m, n+m+1, \ldots, \} \subseteq \text{spec}(\phi).$$
2. If $n + 2^n R(m) \notin \text{spec}(\phi)$ then
$$\text{spec}(\phi) \subseteq \{0, 1, 2, \ldots, n, n+1, n+2, \ldots, n+2^n R(m) - 1\}.$$

Proof of Claim 2:
1) Since $N_0 \geq n + 2^n R(m) \in \text{spec}(\phi)$ there exists $G = (V, E)$, a graph on N_0 vertices, where ϕ is true. Let $x_1, \ldots, x_n$ be vertices such that the following is true of G:

$$(\forall y_1) \cdots (\forall y_m)[\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)].$$

Let $X = \{x_1, \ldots, x_n\}$ and $U = V - X$. Note that $|U| \geq 2^n R(m)$. Map every $u \in U$ to $(b_1, \ldots, b_n) \in \{0,1\}^n$ such that

$$b_i = \begin{cases} 0 \text{ if } (u, x_i) \notin E \\ 1 \text{ if } (u, x_i) \in E \end{cases} \tag{1}$$

Hence every $u \in U$ is mapped to a description of how it relates to every element in X. Since $|U| \geq 2^n R(m)$ there exists $R(m)$ vertices that map to the same vector. Apply Ramsey's theorem to these $R(m)$ vertices to obtain $z_1, \ldots, z_m$ such that the following are true.

- Either the z_i's form a clique or the z_i's form an ind. set. We will assume the z_i's form a clique (the other case is similar).
- All of the z_i's map to the same vector. Hence they all look the same to $x_1, \ldots, x_n$.

Let H_0 be the graph restricted to $X \cup \{z_1, \ldots, z_m\}$. By Claim 1.a H_0 satisfied ϕ. For every $p \geq 1$ we form a graph $H_p = (V_p, E_p)$ on $n + m + p$ vertices that satisfies ϕ.

- $V_p = X \cup \{z_1, \ldots, z_m, z_{m+1}, \ldots, z_{m+p}\}$ where $z_{m+1}, \ldots, z_{m+p}$ are new vertices.
- E_p is the union of the following edges.
  - The edges in H_0,

- For all $1 \le i < j \le n+m+p$ put an edge between z_i and z_j. (If $i, j \le m$ then there is already an edge there.)
- Let $(b_1, \ldots, b_n)$ be the vector that all of the elements of $\{z_1, \ldots, z_m\}$ mapped to. For $m+1 \le j \le m+p$, for $1 \le i \le m$ such that $b_i = 1$, put an edge between z_j and x_i.

As far as X is concerned, all of the $z_1, \ldots, z_{m+p}$ look the same. Hence any subset of the $\{z_1, \ldots, z_{m+p}\}$ of size m will look just like $z_1, \ldots, z_m$ as far as both X is concerned and as far as their connectivity to each other. Hence H_p satisfies ϕ. Hence $n+m+p \in \mathrm{spec}(\phi)$.

2) Assume, by way of contradiction, that some $N_0 > n+2^n R(m) \in \mathrm{spec}(\phi)$. Then, by part 1 of this claim, all $N \ge n+m$ are in $\mathrm{spec}(\phi)$. In particular $n+2^n R(m) \in \mathrm{spec}(\phi)$. This is a contradiction.
End of Proof of Claim 2

We can now give an algorithm for this problem:

1. Input ϕ which begins $(\exists x_1)\cdots(\exists x_n)(\forall y_1)\cdots(\forall y_m)$.
2. Determine if $n+2^n R(m) \in \mathrm{spec}(\phi)$.
   (a) If YES then by Claim 2a
   $$\{n+m, n+m+1, \ldots\} \subseteq \mathrm{spec}(\phi).$$
   For $0 \le i \le n+m-1$ test if $i \in \mathrm{spec}(\phi)$. We now know the finite set of numbers that are not in $\mathrm{spec}(\phi)$. Call this set NOT. Output $\mathrm{spec}(\phi)$ *is* $\mathsf{N} - NOT$.. Note that $\mathrm{spec}(\phi)$ is cofinite.
   (b) if NO then, by Claim 2b
   $$\mathrm{spec}(\phi) \subseteq \{n+1, n+2, \ldots, n+2^n R(m)\}.$$
   Determine, for each N in this set, which ones are in $\mathrm{spec}(\phi)$. Output that finite set.

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