1. (0 points) What is your name? Write it clearly.

2. (40 points) Find polynomials $X_0(n)$, $X_1(n)$, $X_2(n)$ and $X_3(n)$ so that
   (1) they are monotone increasing, (2) polynomial $X_i(n)$, on inputs $i \equiv i \pmod{4}$, output an element of $\mathbb{N}$, (3) THE FOUR STATEMENTS below are true. The polynomials must be expressed so the coefficients are clear, something like:

   $$\frac{87n^2}{13} + \frac{163n}{85} + \frac{17n}{35} - \frac{31}{14}$$

   (This is not close to the answer and the answer need not be quadratic.)

   You need to DERIVE the answer, not just write down polys which happens to work.

   Advice Use Wolfram Alpha.

   THE FOUR STATEMENTS:

   • Let $n \equiv 0 \pmod{4}$. For all 2-colorings of the edges of $K_n$ there exists $\geq X_0(n)$ monochromatic $K_3$’s.
   • Let $n \equiv 1 \pmod{4}$. For all 2-colorings of the edges of $K_n$ there exists $\geq X_1(n)$ monochromatic $K_3$’s.
   • Let $n \equiv 2 \pmod{4}$. For all 2-colorings of the edges of $K_n$ there exists $\geq X_2(n)$ monochromatic $K_3$’s.
   • Let $n \equiv 3 \pmod{4}$. For all 2-colorings of the edges of $K_n$ there exists $\geq X_3(n)$ monochromatic $K_3$’s.

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**SOLUTION** \( R, B, M, ZAN \) are as in the lecture (on the slides).

Let \( COL \) be a 2-coloring of \( K_n \) where we will break down cases based on what \( n \) is congruent to mod 4 later.

As in the proof in class we MAP ZAN to M. As in class the map is 2-to-1, \( |M| = |ZAN|/2 \).

We need to bound \( |ZAN| \)

**Case 1:** \( n \) is odd. Then degree of each vertex is \( n - 1 \equiv 0 \) (mod 2). To maximize \( |ZAN| \) we would, at each vertex, color half of the edges RED and half BLUE. So each vertex contributes \( \left(\frac{n-1}{2}\right)^2 \), and there are \( n \) vertices, so we have \( |ZAN| \leq \frac{(n-1)^2}{4} \).

Hence \( |M| = |ZAN|/2 \leq \frac{(n-1)^2}{8} \).

Since \( |M| \in \mathbb{N} \) we have

\[
|M| \leq \left\lfloor \frac{(n-1)^2}{8} \right\rfloor.
\]

Since \( |R| + |B| + |M| = \binom{n}{3} = \frac{n(n-1)(n-2)}{6} \) we have

\[
|R| + |B| = \frac{n(n-1)(n-2)}{6} - M \geq \frac{n(n-1)(n-2)}{6} - \left\lfloor \frac{(n-1)^2}{8} \right\rfloor
\]

If \( n \equiv 1 \) (mod 4) then \( n - 1 \equiv 0 \) (mod 4), so \( (n-1)^2 \equiv 0 \) (mod 16), so

\[
\left\lfloor \frac{(n-1)^2}{8} \right\rfloor = \frac{(n-1)^2}{8}.
\]

Using Wolfram Alpha we get that this is

\[
X_1(n) = \frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24}
\]

We omit the \( X_3(n) \) case.

**Case 2:** \( n \) is even. Then degree of each vertex is \( n - 1 \) which is odd. To maximize \( |ZAN| \) we would, at each vertex, color \( \frac{n-2}{2} \) of the edges
RED and color $\frac{n}{2}$ of the edges BLUE. So each vertex contributes $\frac{n(n-2)}{4}$, and there are $n$ vertices, so we have $|ZAN| \leq \frac{n^2(n-2)}{4}$.

Hence $|M| = |ZAN|/2 \leq \frac{n^2(n-2)}{8}$.

Since $n \equiv 0 \pmod{2}$, $n^2(n-2) \equiv 0 \pmod{8}$, so the bound on $|M|$ is always in $\mathbb{N}$ (so no need for a floor).

$|M| \leq \frac{n^2(n-2)}{8}$.

Since $R + B + M = \binom{n}{3} = \frac{n(n-1)(n-2)}{6} \leq \frac{n^2(n-2)}{8}$ we have

$$|R| + |B| = \frac{n(n-1)(n-2)}{6} - M \geq \frac{n(n-1)(n-2)}{6} - \frac{n^2(n-2)}{8}.$$

We omit the rest.

END OF SOLUTION

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3. (30 points- 15 points each)

(a) Give sentence \( \phi \) in the lang of graphs with \( \text{SPEC}(\phi) = \{100\} \).

(b) Give sentence \( \phi \) in the lang of graphs with \( \text{SPEC}(\phi) = \{0, 3, 6, \ldots\} \).

**SOLUTION**

(a) Give sentence \( \phi \) in the lang of graphs with \( \text{SPEC}(\phi) = \{100\} \).

**ANSWER:** Recall the convention that if we have \((\exists x_1)(\exists x_2)\) then \( x_1 \neq x_2 \). Hence the following sentence works

\[
(\exists x_1)(\exists x_2) \cdots (\exists x_{100})(\forall y) \left[ \bigvee_{i=1}^{100} y = x_i \right].
\]

(b) Give sentence \( \phi \) in the lang of graphs with \( \text{SPEC}(\phi) = \{0, 3, 6, \ldots\} \).

**ANSWER:** We force the graph to be a union of triangles

\[
(\forall x)(\exists y, z)(\forall w \neq x, y, z) \left[ E(x, y) \land E(x, z) \land E(y, z) \land \neg E(x, w) \right]
\]

**END OF SOLUTION**

4. (30 points) We are looking at the language of colored \( \leq 3 \)-ary hypegraphs. Hence we have the following predicates:

- \( \text{RED}(x), \text{BLUE}(x) \). (so single vertices can be colored RED or BLUE or not at all).
- \( \text{RRED}(x, y), \text{BBLUE}(x, y), \text{GGREEN}(x, y) \). (so 2-ary edges colored RRED or BBLUE or GGREEN or not at all).
- \( \text{RRRED}(x, y, z), \text{BBBLUE}(x, y, z) \). (so 3-ary edges colored RRRED or BBBLUE or not at all).
So a sample sentence is

\[(\exists x_1, x_2, x_3, x_4)(\forall y)[RED(x_1) \land BLUE(x_2) \implies BBBLUE(x_1, x_2, y)]\]

Show that the following is decidable:

Given a \(E^*A^*\) sentence, return spec(\(\phi\)) (which will always be finite or cofinite).

You can assume the standard hypergraph Ramsey Theorem, but aside from that you must prove it from scratch. Here is our Litmus test: Someone in this class who missed my lecture on this should be able to read your proof and understand it.