Project 1, Morally Due March 31 3:30PM (No Dead Cat)

COURSE WEBSITE: `http://www.cs.umd.edu/~gasarch/COURSES/858/S20`

1 Conjecture

In this project, we will consider the following conjecture:

For any $c \in \mathbb{N}$, there exists a number $E = E(c)$ such that for all c-colorings of $\{1, 2, 3, \ldots, E\}$, there exists x, y, z such that:

- *x, y, z are the same color (I bet you saw that coming!), and*
- *$x^2 + y^2 = z^2$*

The conjecture is known to be true for $c = 1$ (this is trivial) and for $c = 2$ (this is not so trivial).

We will gather evidence for how big E might be.

2 Greedy Algorithm

To find lower bounds on $E(c)$, we find a number n and a c-coloring of $[c] = \{1, 2, 3, \ldots, n\}$ such that there is no monochromatic triple x, y, z such that $x^2+y^2 = z^2$. We will call such a c-coloring of $\{1, 2, \ldots, n\}$ a *valid coloring*. If a *valid c-coloring* exists for $\{1, 2, \ldots, n\}$ we will say that $[n]$ *can be c-colored.*

We will consider the following greedy algorithm for finding valid colorings:

For each number k starting from 1, color k with the least color possible. That is, assign k the *least* color χ from the set

$$\{\chi : (\forall x, y < k \text{ s.t. } \mathrm{COL}(x) = \mathrm{COL}(y) = \chi)[x^2 + y^2 \neq k^2]\}$$

Keep coloring points as long as possible, until you reach a number y that can't be colored without creating a monochromatic x, y, z with $x^2 + y^2 = z^2$.

For example, this approach would end up coloring $\mathrm{COL}(1) = \mathrm{COL}(2) = \mathrm{COL}(3) = \mathrm{COL}(4) = 1$; then coloring $\mathrm{COL}(5) = 2$ to avoid $3, 4, 5$ all being the same color.

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3 Randomized (Greedy) Algorithm

Consider the following modification to the Greedy algorithm from the last section: When coloring a number k, consider *all* the valid colors available for color k, and pick one of these *at random*.

That is, *randomly* pick one color from the set

$$\{\chi : (\forall x, y < k \text{ s.t. } \mathrm{COL}(x) = \mathrm{COL}(y) = \chi)[x^2 + y^2 \neq k^2]\}$$

and assign k this color.

As with the original greedy, we continue until some number can't be colored (i.e. the set of valid colors above is empty).

4 The Project

1. (a) Write a program that implements the regular greedy algorithm and outputs a number n, and a coloring of $\{1, 2, \ldots, n\}$.

   (b) Write a program that implements the *randomized* greedy algorithm. It will also output a number n and a coloring of $\{1, 2, \ldots, n\}$.

   Submit your code.

2. Run the greedy algorithm for $c = 2$ to find a number n and a 2-coloring of $\{1, 2, \ldots, n\}$ with no x, y, z such that $x^2 + y^2 = z^2$ and x, y, z are all the same color.

   What number n does the greedy algorithm find and output?

   Run the randomized greedy algorithm for $c = 2$ to find a number n such that $[n]$ can be 2-colored. Run it 50 times. What is the largest n that the randomized greedy algorithm output?

3. Run the greedy algorithm for $c = 3$. What number n does the greedy algorithm find and output?

   Run the randomized greedy algorithm 50 times, for $c = 3$. What is the largest n that the randomized greedy algorithm output?

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4. Run the greedy algorithm for $c = 4$. What number n does the greedy algorithm find and output?

   Run the randomized greedy algorithm 50 times, for $c = 4$. What is the largest n that the randomized greedy algorithm output?

5. Run the greedy algorithm for $c = 5$. What number n does the greedy algorithm find and output?

   Run the randomized greedy algorithm 50 times, for $c = 5$. What is the largest n that the randomized greedy algorithm output?

   (Warning: Nathan's code took 3 minutes to run 50 times, so yours might take a little while as well)