Lower Bounds on $W(3,c)$

Exposition by William Gasarch

May 5, 2020

VDW's Theorem

Theorem (VDW) For all k, c there exists $W = W(k, c)$ such that, for all c-colorings of $[W]$, there exists a, d such that

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- Gowers proved

$$W(k,c) \le 2^{2^{c^{2^{2^{k+9}}}}}$$

Proof uses very hard math.

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k	2 colors	3 colors	4 colors
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- **Idea** Use ML to find VDW numbers.

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Want lower bounds to see how close they are to upper bounds.

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Our Approach

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Our Approach Given V, find c such that there is a c-coloring of $[V]$ with no mono 3-AP's. Try to make c as small as possible.

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Definition $A \subseteq [V]$ is **3-free** if there are no 3-AP's in A. Note that if $[V]$ is colored and has no 3-AP's then every color is 3-free.

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Shifting A If $A \subseteq [V]$ and $t \in [V]$ then

$$A + t = \{x + t \pmod W) : x \in A\}$$

$A + t$ is a **shift of A**.
t is called **the shift**.

The Ideal World is Almost True!

Ideal World 3-free $A \subseteq [V]$ can be shifted around so that none of the shifts overlap. This would be $\frac{V}{|A|}$ shifts and hence there is a $\frac{V}{|A|}$-coloring with no mono 3-AP's.

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What is Prob that some element of $[V]$ was NOT covered?
Let $x \in [V]$ and t be a random shift.
$\Pr(x \in A+t) = \frac{|A|}{V}$.
$\Pr(x \notin A+t) = 1 - \frac{|A|}{V} \sim e^{-|A|/V}$
$\Pr(x \notin A+t_1 \cup \cdots \cup A+t_c) \leq\sim e^{-|A|c/V}$.
$\Pr(\exists x \notin A+t_1 \cup \cdots \cup A+t_c) \leq\sim Ve^{-|A|c/V}$.
We choose c so that this is < 1. $c = \frac{V\ln(V)}{|A|}$
Note $\frac{V\ln(V)}{|A|}$ is close to the ideal of $\frac{V}{|A|}$.

Recap

We have shown the following.

Theorem Let $V \in \mathbb{N}$ and let $A \subseteq [V]$ be a 3-free set. Let $c = \frac{V\ln(V)}{|A|}$. Then there is a c-coloring of $[V]$ with no mono 3-APs. Hence $W(3, c) \geq V$.

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Not so Fast We need to find 3-free sets.

3-Free Set

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3-Free Set Facts

- If A is not 3-free then there exists $a, a+d, a+2d \in A$.
- If A is not 3-free then there exists $x, y, z \in A$ such that $x + z = 2y$.
- **Notation** The size of the largest 3-free set of $[V]$ is denoted $\mathrm{sz}(V)$.

$\mathrm{sz}(V) \geq V^{0.63}$

View $[V]$ as numbers in base 3.

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Size of A $[V]$ in base 3 takes $\log_3(V)$ digits. So

$$|A| \sim 2^{\log_3(V)} \sim V^{\log_3(2)} = V^{0.63}$$

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Shucky Darns! Need to add one more condition.

The Real Set A

A is the set of all $w \in [V]$ such that

- Base 5 rep of w only has 0's, 1's, 2's.
- Base 5 rep of w exactly $1/3$ of the digits are 0.

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Key For all other places $x_i \neq 0$, $z_i \neq 0$.

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SECOND look at the places where $y_i = 1$. $x_i + z_i = 2$ and $x_i \neq 0$, $y_i \neq 0$ Hence $x_i = z_i = 1$.

THIRD look at the places where $y_i = 2$. $x_i + z_i = 4$, so $x_i = z_i = 2$.

The Real Set A

A is the set of all $w \in [V]$ such that

- Base 5 rep of w only has 0's, 1's, 2's.
- Base 5 rep of w exactly $1/3$ of the digits are 0.

3-free

$x = x_L \cdots x_0$

$z = z_L \cdots z_0$

$y = y_L \cdots y_0$

If $x + z = 2y$ then, for all i, $x_i + z_i = 2y_i$.

FIRST look at the $L/3$ places where $y_i = 0$. Then $x_i = z_i = 0$.
Key For all other places $x_i \neq 0$, $z_i \neq 0$.

SECOND look at the places where $y_i = 1$. $x_i + z_i = 2$ and $x_i \neq 0$, $y_i \neq 0$ Hence $x_i = z_i = 1$.

THIRD look at the places where $y_i = 2$. $x_i + z_i = 4$, so $x_i = z_i = 2$.

So $x = y = z$.

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Leave it to the reader to work it out.

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Let r be such that $2^{r(r+1)/2} - 1 \leq V \leq 2^{(r+1)(r+2)/2} - 1$.
Note that $r \sim \sqrt{2\lg(V)}$.

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We denote the ith block as B_i, a number.

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$B_1 = 1$
$B_2 = 3$
$B_3 = 1$
$B_4 = 5$

The Set A

A is the set of all $B_r B_{r-1} \cdots B_1$ such that:

1. For $1 \le i \le r-2$ the leftmost bit of B_i is 0. This leads to carry-free addition.
2. $\sum_{i=1}^{r-2} B_i^2 = B_r B_{r-1}$ (The $B_r B_{r-1}$ is concatenation.)

We leave it to the reader to prove that $|A|$ is as big as we said (this is easy) and that the set is 3-free (This requires some thought.)

Back to $W(3, c)$

Recall that we prove:

Thm Let $V \in \mathbb{N}$ and let $A \subseteq [V]$ be a 3-free set. Then there is a $\frac{V \ln(V)}{|A|}$-coloring of $[V]$ with no mono 3-APs. Hence $W(3, \frac{V \ln(V)}{|A|}) \geq V$.

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Combine these two to get:

Thm Let $V \in \mathbb{N}$. Then there is a $V^{\frac{1}{\sqrt{\lg V}}} \ln(V)$-coloring of $[V]$ with no mono 3-APs. Hence

$$W(3, V^{\frac{1}{\sqrt{\lg V}}} \ln(V)) \geq V.$$