Lower Bounds on $W(3, c)$

Exposition by William Gasarch

May 5, 2020
VDW’s Theorem

**Theorem (VDW)** For all $k, c$ there exists $W = W(k, c)$ such that, for all $c$-colorings of $[W]$, there exists $a, d$ such that

$$a, a + d, \ldots, a + (k - 1)d$$

are the same color.
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- Gowers proved

$$W(k, c) \leq 2^{2^22^k+9}$$

Proof uses very hard math.
### The Only Known VDW Numbers

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Recap

Upper bounds are Ginormous!

Actual numbers are small!

Want lower bounds to see how close they are to upper bounds.
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The Usual Approach

Given $c$, find $W$ such that there is a $c$-coloring of $W$ with no mono 3-AP's. Try to make $W$ as big as possible. We won't be doing that. We do it backwards.
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**Our Approach** Given $V$, find $c$ such that there is a $c$-coloring of $[V]$ with no mono 3-AP’s. Try to make $c$ as small as possible.
3-free Sets

**Definition** $A \subseteq [V]$ is **3-free** if there are no 3-AP’s in $A$. Note that if $[V]$ is colored and has no 3-AP’s then every color is 3-free.
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Idea Find a large subset of $[V]$ with no 3-AP’s. Color it RED!
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Okay... Now what?

**Shifting** $A$ If $A \subseteq [V]$ and $t \in [V]$ then

$$A + t = \{x + t \pmod{W} : x \in A\}$$

$A + t$ is a **shift of $A$**.

$t$ is called **the shift**.
The Ideal World is Almost True!

**Ideal World** 3-free $A \subseteq [V]$ can be shifted around so that none of the shifts overlap. This would be $\frac{V}{|A|}$ shifts and hence there is a $\frac{V}{|A|}$-coloring with no mono 3-AP’s.
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We Use Randomness

We take $c$ random shifts where we determine $c$ later. What is Prob that some element of $[V]$ was NOT covered?
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Let $x \in [V]$ and $t$ be a random shift.
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Let $x \in [V]$ and $t$ be a random shift.
$\Pr(x \in A + t) = \frac{|A|}{V}$.
We take $c$ random shifts where we determine $c$ later. What is Prob that some element of $[V]$ was NOT covered? Let $x \in [V]$ and $t$ be a random shift.

$$\Pr(x \in A + t) = \frac{|A|}{V}.$$  

$$\Pr(x \notin A + t) = 1 - \frac{|A|}{V} \sim e^{-|A|/V}.$$

$$\Pr(x \notin A + t_1 \cup \cdots \cup A + t_c) \leq \sim e^{-|A|c/V}.$$  

$$\Pr(\exists x \notin A + t_1 \cup \cdots \cup A + t_c) \leq \sim Ve^{-|A|c/V}.$$  

We choose $c$ so that this is $< 1$. $c = \frac{V \ln(V)}{|A|}$

**Note** $\frac{V \ln(V)}{|A|}$ is close to the ideal of $\frac{V}{|A|}$.
Recap

We have shown the following.

**Theorem** Let $V \in \mathbb{N}$ and let $A \subseteq [V]$ be a 3-free set. Let $c = \frac{V \ln(V)}{|A|}$. Then there is a $c$-coloring of $[V]$ with no mono 3-APs. Hence $W(3, c) \geq V$. 

So, we’re done!

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3-Free Set

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3-Free Set Facts

- If $A$ is not 3-free then there exists $a, a + d, a + 2d \in A$.
- If $A$ is not 3-free then there exists $x, y, z \in A$ such that $x + z = 2y$.
- **Notation** The size of the largest 3-free set of $[V]$ is denoted $sz(V)$. 
$\text{sz}(V) \geq V^{0.63}$

View $[V]$ as numbers in base 3.

$$A = \{ w \in [V] : \text{Base 3 rep of } w \text{ only has 0's and 1's} \}$$
\[ \text{sz}(V) \geq V^{0.63} \]

View \([V]\) as numbers in base 3.

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**3-Free** Assume \(x, y, z \in A\) and \(x + z = 2y\).

**Key** Since base 3 rep of \(x, y, z\) has only 0's and 1's, adding them is carry free.

\[ x = x_L \cdots x_0 \]
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Size of $A$ $[V]$ in base 3 takes $\log_3(V)$ digits. So

$$|A| \sim 2^{\log_3(V)} \sim V^{\log_3(2)} = V^{0.63}$$
\( \text{sz}(V) \geq V^{0.68} \)

View \([V]\) as numbers in base 5.
(Attempt- it won’t work)

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A = \{ w \in [V] : \text{Base 5 rep of } w \text{ only has 0’s, 1’s, 2’s} \}
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**3-Free** Assume \( x, y, z \in A \) and \( x + z = 2y \).

**Key** Since base 5 rep of \( x, y, z \) has only 0’s, 1’s, 2’s adding them is carry free.

\( x = x_L \cdots x_0 \)
\( z = z_L \cdots z_0 \)
\( y = y_L \cdots y_0 \)

If \( x + z = 2y \) then, for all \( i \), \( x_i + z_i = 2y_i \).
If \( y_i = 0 \) the then \( x_i = z_i = 0 \).
If \( y_i = 1 \) the then \( x_i = z_i = 1 \).
$\text{sz}(V) \geq V^{0.68}$

View $[V]$ as numbers in base 5.
(Attempt- it won’t work)

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Shucky Darns! Need to add one more condition.
The Real Set $A$

$A$ is the set of all $w \in \mathbb{V}$ such that

- Base 5 rep of $w$ only has 0’s, 1’s, 2’s.
- Base 5 rep of $w$ exactly 1/3 of the digits are 0.

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FIRST look at the $L/3$ places where $y_i = 0$. Then $x_i = z_i = 0$.

Key For all other places $x_i \neq 0$, $z_i \neq 0$. 
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3-free

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THIRD look at the places where \( y_i = 2 \). \( x_i + z_i = 4 \), so \( x_i = z_i = 2 \).
The Real Set $A$

$A$ is the set of all $w \in [V]$ such that

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If $x + z = 2y$ then, for all $i$, $x_i + z_i = 2y_i$.

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**SECOND** look at the places where $y_i = 1$. $x_i + z_i = 2$ and $x_i \neq 0$, $y_i \neq 0$ Hence $x_i = z_i = 1$.

**THIRD** look at the places where $y_i = 2$. $x_i + z_i = 4$, so $x_i = z_i = 2$.

So $x = y = z$. 
What is $|A|$?

Choose $L/3$ of the digits to be 0. \[ \binom{L}{L/3} \sim L^{L/3} \]
What is $|A|$?

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For the remainder use 1’s or 2’s, so $2^{2L/3}$
What is $|A|$?

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For the remainder use 1’s or 2’s, so $2^{2L/3}$

Leave it to the reader to work it out.
\[ \text{sz}(V) \geq V^{1 - \frac{1}{\sqrt{\lg V}}} \]

Let \( r \) be such that \( 2^{r(r+1)/2} - 1 \leq V \leq 2^{(r+1)(r+2)/2} - 1 \).

Note that \( r \sim \sqrt{2 \lg(V)} \).
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Write the numbers in \([V]\) in base 2.

Break the numbers into \( r \) blocks of bits.
Let $r$ be such that \( 2^{r(r+1)/2} - 1 \leq V \leq 2^{(r+1)(r+2)/2} - 1 \). Note that $r \sim \sqrt{2 \lg(V)}$.

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The first (rightmost) block is one 1 long.
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Break the numbers into $r$ blocks of bits.

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Write the numbers in $[V]$ in base 2.

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The first (rightmost) block is one 1 long.

The second block is 2 bits long.

The $r$th block is $r$ bits long.

We denote the $i$th block as $B_i$, a number.
An Example!

991746118991 in binary is

1110011011101000101011001101010101001111

B_1 = 1
B_2 = 3
B_3 = 1
B_4 = 5
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We write it as:

000001110; 01101110; 1000101; 011001; 10101; 0101; 001; 11; 1
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The Set $A$

$A$ is the set of all $B_rB_{r-1}\cdots B_1$ such that:

1. For $1 \leq i \leq r - 2$ the leftmost bit of $B_i$ is 0. This leads to carry-free addition.

2. $\sum_{i=1}^{r-2} B_i^2 = B_rB_{r-1}$ (The $B_rB_{r-1}$ is concatenation.)

We leave it to the reader to prove that $|A|$ is as big as we said (this is easy) and that the set is 3-free (This requires some thought.)
Recall that we prove:

**Thm** Let $V \in \mathbb{N}$ and let $A \subseteq [V]$ be a 3-free set. Then there is a $\frac{V \ln(V)}{|A|}$-coloring of $[V]$ with no mono 3-APs. Hence $W(3, \frac{V \ln(V)}{|A|}) \geq V$. 

**Back to $W(3, c)$**
Recall that we prove:

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Recall that we sketched:

Thm There exists a 3-free subset of $[V]$ of size $\geq V^{1 - \frac{1}{\sqrt{\lg V}}}$
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Recall that we sketched:

**Thm** There exists a 3-free subset of $[V]$ of size $\geq V^{1 - \frac{1}{\sqrt{\lg V}}}$

Combine these two to get:

**Thm** Let $V \in \mathbb{N}$. Then there is a $V^{\frac{1}{\sqrt{\lg V}} \ln(V)}$-coloring of $[V]$ with no mono 3-APs. Hence

$$W(3, V^{\frac{1}{\sqrt{\lg V}} \ln(V)}) \geq V.$$