Problems with a Point: Exploring Math and Computer Science

April 19, 2020
Authors: William Gasarch Clyde Kruskal

April 19, 2020
How This Book Came to Be

April 19, 2020
In 2003 Lance Fortnow started Complexity Blog

In 2007 Bill Gasarch joined and it was a co-blog.

In 2015 various book publishers asked us

Can you make a book out of your blog?

Lance declined but Bill said YES.
Bill took the posts that had the following format:

- make a point about mathematics
- do some math to underscore those points

and made those into chapters.
Bill took the posts that had the following format:

- make a point **about** mathematics
- do some math to **underscore** those points

and made those into chapters.

**Caveat:** Not every chapter is quite like that.
To quote Ralph Waldo Emerson

*A foolish consistency is the hobgoblin of small minds.*
Possible Subtitles

Problems with a Point needed a subtitle.
I proposed
Possible Subtitles

Problems with a Point needed a subtitle.
I proposed
Problems with a Point: Mathematical Musing and Math to make those Musings Magnificent
Problems with a Point needed a subtitle.  
I proposed

**Problems with a Point: Mathematical Musing and Math to make those Musings Magnificent**

The publisher said **NO!**
Possible Subtitles

Problems with a Point needed a subtitle.
I proposed
Problems with a Point: Mathematical Musing and Math to make those Musings Magnificent

The publisher said NO!

I proposed
Problems with a Point: Mathematical Meditations and Computer Science Cogitations
Possible Subtitles

Problems with a Point needed a subtitle.
I proposed
Problems with a Point: Mathematical Musing and Math to make those Musings Magnificent

The publisher said NO!

I proposed
Problems with a Point: Mathematical Meditations and Computer Science Cogitations

The publisher said NO!
Possible Subtitles

Problems with a Point needed a subtitle.
I proposed
Problems with a Point: Mathematical Musing and Math to make those Musings Magnificent

The publisher said NO!

I proposed
Problems with a Point: Mathematical Meditations and Computer Science Cogitations

The publisher said NO!

The publisher wisely decided to be less cute and more informative:
Problems with a Point: Exploring Math and Computer Science
Clyde Joins the Project!

After some samples of Bill’s writing the publisher said
Clyde Joins the Project!

After some samples of Bill’s writing the publisher said

Please Procure People to Polish Prose and Proofs of Problems with a Point

so
Clyde Joins the Project!

After some samples of Bill’s writing the publisher said

**Please Procure People to Polish Prose and Proofs of Problems with a Point**

so

Clyde Kruskal became a co-author.
Clyde Joins the Project!

After some samples of Bill’s writing the publisher said

**Please Procure People to Polish Prose and Proofs of Problems with a Point**

so

Clyde Kruskal became a co-author.

Now onto some samples of the book!
Point: Students Can Give Strange Answers

April 19, 2020
The Paint Can Problem

From the Year 2000 Maryland Math Competition: 
*There are 2000 cans of paint. Show that at least one of the following two statements is true:*

▶ There are at least 45 cans of the same color.
▶ There are at least 45 cans that are different colors.

Work on it.
The Paint Can Problem

From the Year 2000 Maryland Math Competition:

There are 2000 cans of paint. Show that at least one of the following two statements is true:

▶ There are at least 45 cans of the same color.
▶ There are at least 45 cans that are different colors.

Work on it.

Answer:
If there are 45 different colors of paint then we are done. Assume there are \( \leq 44 \) different colors. If all colors appear \( \leq 44 \) times then there are \( 44 \times 44 = 1936 < 2000 \) cans of paint, a contradiction.

Note: this was Problem 1, which is supposed to be easy and indeed 95% got it right. What about the other 5%? Next slide.
There are 2000 cans of paint. Show that at least one of the following two statements is true:

- There are at least 45 cans of the same color.
- There are at least 45 cans that are different colors.
One of the Wrong Answers. Or is it?

There are 2000 cans of paint. Show that at least one of the following two statements is true:

- There are at least 45 cans of the same color.
- There are at least 45 cans that are different colors.

**ANSWER:**

Paint cans are grey. Hence there are all the same color. Therefore there are 2000 cans that are the same color.
There are 2000 cans of paint. Show that at least one of the following two statements is true:

▶ There are at least 45 cans of the same color.
▶ There are at least 45 cans that are different colors.

**ANSWER:**

*Paint cans are grey. Hence there are all the same color. Therefore there are 2000 cans that are the same color.*

What do you think:

▶ Thats just stupid. 0 points.
▶ Question says *cans of the same color.*... The full 30 pts.
▶ Not only does he get 30 points, but everyone else should get 0.
Another Wrong Answers. Or is it?

There are 2000 cans of paint. Show that at least one of the following two statements is true:

- There are at least 45 cans of the same color.
- There are at least 45 cans that are different colors.

If you look at a paint color really really carefully there will be differences. Hence, even if two cans seem to both be (say) RED, they are really different. Therefore there are 2000 cans of different colors.

What do you think:
- Thats just stupid. 0 points.
- Well...
  - he's got a point. 30 points in fact.
- Not only does he get 30 points, but everyone else should get 0.
Another Wrong Answers. Or is it?

There are 2000 cans of paint. Show that at least one of the following two statements is true:

▶ There are at least 45 cans of the same color.
▶ There are at least 45 cans that are different colors.

**ANSWER:**
If you look at a paint color really really carefully there will be differences. Hence, even if two cans seem to both be (say) RED, they are really different. Therefore there are 2000 cans of different colors.
Another Wrong Answers. Or is it?

There are 2000 cans of paint. Show that at least one of the following two statements is true:

▶ There are at least 45 cans of the same color.
▶ There are at least 45 cans that are different colors.

**ANSWER:**
If you look at a paint color really really carefully there will be differences. Hence, even if two cans seem to both be (say) RED, they are really different. Therefore there are 2000 cans of different colors.

What do you think:
▶ Thats just stupid. 0 points.
▶ Well... he’s got a point. 30 points in fact.
▶ Not only does he get 30 points, but everyone else should get 0.
A Triangle Problem

From the year 2007 Maryland Math Competition.

QUESTION: Let $ABC$ be a fixed triangle. Let $COL$ be any 2-coloring of the plane where each point is colored with red or green. Prove that there is a triangle $DEF$ in the plane such that $DEF$ is similar to $ABC$ and the vertices of $DEF$ all have the same color.
A Triangle Problem

From the year 2007 Maryland Math Competition.

QUESTION: Let $ABC$ be a fixed triangle. Let $COL$ be any 2-coloring of the plane where each point is colored with red or green. Prove that there is a triangle $DEF$ in the plane such that $DEF$ is similar to $ABC$ and the vertices of $DEF$ all have the same color.

Note I think I was assigned to grade it since it looks like the kind of problem I would make up, even though I didn’t. It was problem 5 (out of 5) and was hard. About 100 students tried it, 8 got full credit, 10 got partial credit.
QUESTION: Let ABC be a fixed triangle. Let COL be any 2-coloring of the plane where each point is colored with red or green. Prove that there is a triangle DEF in the plane such that DEF is similar to ABC and the vertices of DEF all have the same color.
QUESTION: Let ABC be a fixed triangle. Let COL be any 2-coloring of the plane where each point is colored with red or green. Prove that there is a triangle DEF in the plane such that DEF is similar to ABC and the vertices of DEF all have the same color.

Funny Answer One:
All the vertices are red because I can make them whatever color I want. I can also write at a 30 degree angle to the bottom of this paper (The students answer was written at a 30 degree angle to the bottom of the paper.) if thats what I feel like doing at the moment. Just like $2 + 2 = 5$ if thats what my math teacher says. Math is pretty subjective anyway.
All the vertices are red because I can make them whatever color I want. I can also write at a 30 degree angle to the bottom of this paper (The students answer was written at a 30 degree angle to the bottom of the paper.) if thats what I feel like doing at the moment. Just like $2 + 2 = 5$ if thats what my math teacher says. Math is pretty subjective anyway.

**Theorem**
The students is not serious.

**Proof**
Assume, by contradiction, that they are serious. Then they really think math is subjective. Hence they don't really understand math. Hence they would not have done well enough on Part I to qualify for Part II. But they took Part II. Contradiction.
Was Student One Serious?

All the vertices are red because I can make them whatever color I want. I can also write at a 30 degree angle to the bottom of this paper (The students answer was written at a 30 degree angle to the bottom of the paper.) if that’s what I feel like doing at the moment. Just like $2 + 2 = 5$ if that’s what my math teacher says. Math is pretty subjective anyway.

**Theorem** The students is not serious.

**Proof** Assume, by contradiction, that they are serious. Then they really think math is subjective. Hence they don’t really understand math. Hence they would not have done well enough on Part I to qualify for Part II. But they took Part II. Contradiction.
QUESTION: Let ABC be a fixed triangle. Let COL be any 2-coloring of the plane where each point is colored with red or green. Prove that there is a triangle DEF in the plane such that DEF is similar to ABC and the vertices of DEF all have the same color.
QUESTION: Let ABC be a fixed triangle. Let COL be any 2-coloring of the plane where each point is colored with red or green. Prove that there is a triangle DEF in the plane such that DEF is similar to ABC and the vertices of DEF all have the same color.

I like to think that we live in a world where points are not judged by their color, but by the content of their character. Color should be irrelevant in the plane. To prove that there exists a group of points where only one color is acceptable is a reprehensible act of bigotry and discrimination.
QUESTION: Let ABC be a fixed triangle. Let COL be any 2-coloring of the plane where each point is colored with red or green. Prove that there is a triangle DEF in the plane such that DEF is similar to ABC and the vertices of DEF all have the same color.

I like to think that we live in a world where points are not judged by their color, but by the content of their character. Color should be irrelevant in the plane. To prove that there exists a group of points where only one color is acceptable is a reprehensible act of bigotry and discrimination.

Was Student Two Serious.
QUESTION: Let ABC be a fixed triangle. Let COL be any 2-coloring of the plane where each point is colored with red or green. Prove that there is a triangle DEF in the plane such that DEF is similar to ABC and the vertices of DEF all have the same color.

I like to think that we live in a world where points are not judged by their color, but by the content of their character. Color should be irrelevant in the plane. To prove that there exists a group of points where only one color is acceptable is a reprehensible act of bigotry and discrimination.

Was Student Two Serious. Yes.
QUESTION: Let ABC be a fixed triangle. Let COL be any 2-coloring of the plane where each point is colored with red or green. Prove that there is a triangle DEF in the plane such that DEF is similar to ABC and the vertices of DEF all have the same color.

I like to think that we live in a world where points are not judged by their color, but by the content of their character. Color should be irrelevant in the plane. To prove that there exists a group of points where only one color is acceptable is a reprehensible act of bigotry and discrimination.

Was Student Two Serious. Yes. About Justice!
Each point in the plane is colored either red or green. Let ABC be a fixed triangle. Prove that there is a triangle DEF in the plane such that DEF is similar to ABC and the vertices of DEF all have the same color.
Fix a 2-coloring of the plane.
There are 3 equally-spaced mono points on $x$-axis

**Proof** Clearly there are two points on the $x$-axis of the same color: $p_1, p_2$ are RED. If $p_3$, the midpoint of $p_1, p_2$, is RED then $p_1, p_3, p_2$ are all RED. DONE. Hence we assume $p_3$ is GREEN.

Let $p_4$ be such that $|p_1 - p_4| = |p_2 - p_1|$. If $p_4$ is RED then $p_4, p_1, p_2$ are all RED. DONE. Hence we assume $p_4$ is GREEN.

Let $p_5$ be such that $|p_5 - p_2| = |p_2 - p_1|$. If $p_5$ is RED then $p_1, p_2, p_5$ are all RED. DONE. Hence we assume $p_5$ is GREEN.

Only case left $p_3, p_4, p_5$ are all GREEN. DONE.
Figure: Triangle Similar to $ABC$ with Monochromatic Vertices

$P, Q, R$ are RED.

If $T$ or $U$ or $S$ are RED then get RED Triangle similar to $ABC$.

If not then ALL of $T, U, S$ are GREEN, so get GREEN triangle similar to $ABC$. 
Bill assigned the following in Discrete Math: For each of the following sequences find a simple function $A(n)$ such that the sequence is $A(1)$, $A(2)$, $A(3)$, ... 

1. 10, -17, 24, -31, 38, -45, 52, ...  
2. -1, 1, 5, 13, 29, 61, 125, ...  
3. 6, 9, 14, 21, 30, 41, 54, ...  

**Caveat:** These are NOT trick questions. Work on it.
Simple Functions

Bill assigned the following in Discrete Math: For each of the following sequences find a simple function $A(n)$ such that the sequence is $A(1), A(2), A(3), \ldots$

1. 10, -17, 24, -31, 38, -45, 52, \ldots
2. -1, 1, 5, 13, 29, 61, 125, \ldots
3. 6, 9, 14, 21, 30, 41, 54, \ldots

Caveat: These are NOT trick questions.
Work on it.

1. 10, -17, 24, -31, 38, -45, 52, \ldots \quad A(n) = (-1)^{n+1}(7n + 3).
2. -1, 1, 5, 13, 29, 61, 125, \ldots \quad A(n) = 2^n - 3.
3. 6, 9, 14, 21, 30, 41, 54, \ldots \quad A(n) = n^2 + 5.
A Student asks — What is a Simple Function?

One student, in earnest, emailed Bill the following:

*We never defined Simple Function in class so I went to Wikipedia. It said that A Simple Function is a linear combination of indicator functions of measurable sets. Is that what you want us to use?*
One student, in earnest, emailed Bill the following:

We never defined **Simple Function** in class so I went to Wikipedia. It said that **A Simple Function is a linear combination of indicator functions of measurable sets.** Is that what you want us to use?

I doubt the student knows what those terms mean.
A Student asks — What is a Simple Function?

One student, in earnest, emailed Bill the following:

We never defined Simple Function in class so I went to Wikipedia. It said that A Simple Function is a linear combination of indicator functions of measurable sets. Is that what you want us to use?

I doubt the student knows what those terms mean
I doubt Clyde knows what those terms mean.
One student, in earnest, emailed Bill the following:

*We never defined **Simple Function** in class so I went to Wikipedia. It said that **A Simple Function is a linear combination of indicator functions of measurable sets.** Is that what you want us to use?*

I doubt the student knows what those terms mean.
I doubt Clyde knows what those terms mean.
I don’t know what these terms mean.
A Student asks — What is a Simple Function?

One student, in earnest, emailed Bill the following:

_We never defined **Simple Function** in class so I went to Wikipedia. It said that **A Simple Function** is a linear combination of indicator functions of measurable sets. Is that what you want us to use?_

I doubt the student knows what those terms mean.
I doubt Clyde knows what those terms mean.
I don’t know what these terms mean.

I told him NO— all I wanted is an easy-to-describe function. I should have told him to use that definition to see what he came up with.
A Student asks — What is a Simple Function?

One student, in earnest, emailed Bill the following:

*We never defined **Simple Function** in class so I went to Wikipedia. It said that **A Simple Function is a linear combination of indicator functions of measurable sets.** Is that what you want us to use?*

I doubt the student knows what those terms mean.
I doubt Clyde knows what those terms mean.
I don’t know what these terms mean.

I told him NO— all I wanted is an easy-to-describe function. I should have told him to use that definition to see what he came up with.

The student got the first one right, but left the other two blank.
When Do Patterns Hold?

The last question brings up the question of when patterns do and don’t hold. We looked for cases where a pattern *did not* hold.
What is the max number of regions formed by connecting every pair of $n$ points on a circle. For $n = 1, 2, 3, 4, 5$:

Tempted to guess $2^{n-1}$.
First Non-Pattern: $n$ Points on a circle

What is the max number of regions formed by connecting every pair of $n$ points on a circle. For $n = 1, 2, 3, 4, 5$:

Tempted to guess $2^{n-1}$.
But for $n = 6$, the number of regions is only 31.
First Non-Pattern: $n$ Points on a circle

What is the max number of regions formed by connecting every pair of $n$ points on a circle. For $n = 1, 2, 3, 4, 5$:

Tempted to guess $2^{n-1}$.
But for $n = 6$, the number of regions is only 31.
The actual number of regions for $n$ points is $\binom{n}{4} + \binom{n}{2} + 1$. 
Second Non-Pattern: Borwein Integrals

\[ \int_0^{\infty} \frac{\sin x}{x} = \frac{\pi}{2} \]

\[ \int_0^{\infty} \frac{\sin x \sin \frac{x}{3}}{x \frac{x}{3}} = \frac{\pi}{2} \]

\[ \vdots \]

\[ \int_0^{\infty} \frac{\sin x \sin \frac{x}{3} \sin \frac{x}{5} \sin \frac{x}{7} \sin \frac{x}{9} \sin \frac{x}{11} \sin \frac{x}{13}}{x \frac{x}{3} \frac{x}{5} \frac{x}{7} \frac{x}{9} \frac{x}{11} \frac{x}{13}} = \frac{\pi}{2} \]
Second Non-Pattern: Borwein Integrals

\[ \int_{0}^{\infty} \frac{\sin x}{x} = \frac{\pi}{2} \]

\[ \int_{0}^{\infty} \frac{\sin x \sin \frac{x}{3}}{x} = \frac{\pi}{2} \]

\[ \vdots \]

\[ \int_{0}^{\infty} \frac{\sin x \sin \frac{x}{3} \sin \frac{x}{5} \sin \frac{x}{7} \sin \frac{x}{9} \sin \frac{x}{11} \sin \frac{x}{13}}{x} = \frac{\pi}{2} \]

But

\[ \int_{0}^{\infty} \frac{\sin x \sin \frac{x}{3} \sin \frac{x}{5} \sin \frac{x}{7} \sin \frac{x}{9} \sin \frac{x}{11} \sin \frac{x}{13} \sin \frac{x}{15}}{x} = \frac{\pi}{2} \]

\[ \frac{4678079247134407386537864469\pi}{93561584940640907310521750000} \sim 0.999999999852937186 \times \frac{\pi}{2} \]
Why the breakdown at 15?

Because

\[ \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{13} < 1 \]

but

\[ \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{15} > 1. \]

For more Google

Borwein Integral
Computers to FIND proofs vs Computers to DO Proofs

April 19, 2020
Colorings and Square Differences

The following are all true:

1. There exists a number \( W_2 \) such that, for all 2-colorings of \( \{1, \ldots, W_2\} \) there exists 2 nums, square-apart, same color.
The following are all true:

1. There exists a number $W_2$ such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.
2. There exists a number $W_3$ such that, for all 3-colorings of $\{1, \ldots, W_3\}$ there exists 2 nums, square-apart, same color.
The following are all true:

1. There exists a number $W_2$ such that, for all 2-colorings of \{1, \ldots, W_2\} there exists 2 nums, square-apart, same color.

2. There exists a number $W_3$ such that, for all 3-colorings of \{1, \ldots, W_3\} there exists 2 nums, square-apart, same color.

3. There exists a number $W_4$ such that, for all 3-colorings of \{1, \ldots, W_4\} there exists two nums, square-apart, same color.
The following are all true:

1. There exists a number $W_2$ such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.
2. There exists a number $W_3$ such that, for all 3-colorings of $\{1, \ldots, W_3\}$ there exists 2 nums, square-apart, same color.
3. There exists a number $W_4$ such that, for all 3-colorings of $\{1, \ldots, W_4\}$ there exists two nums, square-apart, same color.
4. For all $c$ there exists a number $W_c$ ....
Colorings and Square Differences

The following are all true:

1. There exists a number $W_2$ such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.

2. There exists a number $W_3$ such that, for all 3-colorings of $\{1, \ldots, W_3\}$ there exists 2 nums, square-apart, same color.

3. There exists a number $W_4$ such that, for all 3-colorings of $\{1, \ldots, W_4\}$ there exists two nums, square-apart, same color.

4. For all $c$ there exists a number $W_c$ 

The proofs in the literature of these theorems give enormous bounds on $W_2, W_3, W_4, W_c$. We look at easier proofs with two points in mind:

- Would they make good questions on a HS math competition.
- The role of Computers in these proofs.
2-colorings and Square Differences

There exists a number $W_2$ such that, for all 2-colorings of \{1, \ldots, W_2\} there exists 2 nums, square-apart, same color.
2-colorings and Square Differences

There exists a number $W_2$ such that, for all 2-colorings of \{1, \ldots, W_2\} there exists 2 nums, square-apart, same color.

Think About how to prove it and what $W_2$ is.
There exists a number $W_2$ such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.

Think About how to prove it and what $W_2$ is.

Let COL be a 2-coloring of $\{1, 2, 3, \ldots\}$ with colorings $R$ and $B$. We can assume COL(1) = $R$. 

AH-HA: COL(1) = COL(5) and $5 - 1 = 4 = 2^2$. So $W_2 \leq 5$.

AH-HA: RBRB shows that $W_2 \leq 5$.

So $W_2 = 4$.

Upshot Could be easy HS Math Comp Prob. No computer used.
2-colorings and Square Differences

There exists a number $W_2$ such that, for all 2-colorings of \{1, \ldots, W_2\} there exists 2 nums, square-apart, same color.

Think About how to prove it and what $W_2$ is.

Let $COL$ be a 2-coloring of \{1, 2, 3, \ldots\} with colorings $R$ and $B$. We can assume $COL(1) = R$. Since 1 is a square $COL(2) = B$. 

AH-HA: $COL(1) = COL(5)$ and $5 - 1 = 4 = 2^2$. So $W_2 \leq 5$. 

AH-HA: $RBRB$ shows that $W_2 \leq 5$. 

So $W_2 = 4$. 

Upshot Could be easy HS Math Comp Prob. No computer used.
2-colorings and Square Differences

There exists a number $W_2$ such that, for all 2-colorings of \{1, \ldots, W_2\} there exists 2 nums, square-apart, same color.

Think About how to prove it and what $W_2$ is.

Let COL be a 2-coloring of \{1, 2, 3, \ldots\} with colorings $R$ and $B$. We can assume $\text{COL}(1) = R$.

Since 1 is a square $\text{COL}(2) = B$.

Since 1 is a square $\text{COL}(3) = R$. 

AH-HA: $\text{COL}(1) = \text{COL}(5)$ and $5 - 1 = 4 = 2^2$. So $W_2 \leq 5$.

AH-HA: $RBRB$ shows that $W_2 \leq 5$.

So $W_2 = 4$.

Upshot

Could be easy HS Math Comp Prob. No computer used.
There exists a number $W_2$ such that, for all 2-colorings of $\{1, \ldots, W_2\}$ there exists 2 nums, square-apart, same color.

Think About how to prove it and what $W_2$ is.

Let $\text{COL}$ be a 2-coloring of $\{1, 2, 3, \ldots\}$ with colorings $R$ and $B$. We can assume $\text{COL}(1) = R$.
Since 1 is a square $\text{COL}(2) = B$.
Since 1 is a square $\text{COL}(3) = R$.
Since 1 is a square $\text{COL}(4) = B$. 

AH-HA: $\text{COL}(1) = \text{COL}(5)$ and $5 - 1 = 4 = 2^2$. So $W_2 \leq 5$.
AH-HA: $RBRB$ shows that $W_2 \leq 5$.
So $W_2 = 4$.

Upshot
Could be easy HS Math Comp Prob. No computer used.
2-colorings and Square Differences

There exists a number $W_2$ such that, for all 2-colorings of \{1, \ldots, W_2\} there exists 2 nums, square-apart, same color.

Think About how to prove it and what $W_2$ is.

Let COL be a 2-coloring of \{1, 2, 3, \ldots\} with colorings $R$ and $B$. We can assume COL(1) = $R$.
Since 1 is a square COL(2) = $B$.
Since 1 is a square COL(3) = $R$.
Since 1 is a square COL(4) = $B$.
Since 1 is a square COL(5) = $R$.
2-colorings and Square Differences

There exists a number $W_2$ such that, for all 2-colorings of \{1, \ldots, W_2\} there exists 2 nums, square-apart, same color.

Think About how to prove it and what $W_2$ is.

Let $COL$ be a 2-coloring of \{1, 2, 3, \ldots\} with colorings $R$ and $B$. We can assume $COL(1) = R$. Since 1 is a square $COL(2) = B$. Since 1 is a square $COL(3) = R$. Since 1 is a square $COL(4) = B$. Since 1 is a square $COL(5) = R$.

AH-HA: $COL(1) = COL(5) \text{ and } 5 - 1 = 4 = 2^2$. So $W_2 \leq 5$. 
2-colorings and Square Differences

There exists a number $W_2$ such that, for all 2-colorings of \{1, \ldots, W_2\} there exists 2 nums, square-apart, same color.

Think About how to prove it and what $W_2$ is.

Let COL be a 2-coloring of \{1, 2, 3, \ldots\} with colorings $R$ and $B$. We can assume $\text{COL}(1) = R$.
Since 1 is a square $\text{COL}(2) = B$.
Since 1 is a square $\text{COL}(3) = R$.
Since 1 is a square $\text{COL}(4) = B$.
Since 1 is a square $\text{COL}(5) = R$.
AH-HA: $\text{COL}(1) = \text{COL}(5)$ and $5 - 1 = 4 = 2^2$. So $W_2 \leq 5$.
AH-HA: $RBRB$ shows that $W_2 \leq 5$. 
There exists a number $W_2$ such that, for all 2-colorings of \{1, \ldots, W_2\} there exists 2 nums, square-apart, same color.

Think About how to prove it and what $W_2$ is.

Let COL be a 2-coloring of \{1, 2, 3, \ldots\} with colorings $R$ and $B$. We can assume $COL(1) = R$.
Since 1 is a square $COL(2) = B$.
Since 1 is a square $COL(3) = R$.
Since 1 is a square $COL(4) = B$.
Since 1 is a square $COL(5) = R$.
AH-HA: $COL(1) = COL(5)$ and $5 - 1 = 4 = 2^2$. So $W_2 \leq 5$.
AH-HA: $RBRB$ shows that $W_2 \leq 5$.
So $W_2 = 4$. 
2-colorings and Square Differences

There exists a number $W_2$ such that, for all 2-colorings of \{1, \ldots, W_2\} there exists 2 nums, square-apart, same color.

Think About how to prove it and what $W_2$ is.

Let COL be a 2-coloring of \{1, 2, 3, \ldots\} with colorings $R$ and $B$.
We can assume $\text{COL}(1) = R$.
Since 1 is a square $\text{COL}(2) = B$.
Since 1 is a square $\text{COL}(3) = R$.
Since 1 is a square $\text{COL}(4) = B$.
Since 1 is a square $\text{COL}(5) = R$.

AH-HA: $\text{COL}(1) = \text{COL}(5)$ and $5 - 1 = 4 = 2^2$. So $W_2 \leq 5$.
AH-HA: $RBRB$ shows that $W_2 \leq 5$.
So $W_2 = 4$.

Upshot Could be easy HS Math Comp Prob. No computer used.
3-colorings and Square Differences

In $W_2$-proof had $\text{COL}(1) = \text{COL}(5)$. Need similar for $W_3$.
Let $\text{COL}$ be 3-coloring of $\{1, 2, 3, \ldots\}$, uses $R$, $B$, $G$.
$\text{COL}(1) = R$. 

Figure: $\text{COL}(x) = \text{COL}(x+41)$
3-colorings and Square Differences

In $W_2$-proof had $\text{COL}(1) = \text{COL}(5)$. Need similar for $W_3$.
Let $\text{COL}$ be 3-coloring of $\{1, 2, 3, \ldots\}$, uses $R$, $B$, $G$.
$\text{COL}(1) = R$.

Figure: $\text{COL}(x) = \text{COL}(x + 41)$
Since \( \text{COL}(x) = \text{COL}(x + 41) \ldots \)

Use \( \text{COL}(x) = \text{COL}(x + 41) \) to finish the proof and find upper bound on \( W_3 \).
Since \( \text{COL}(x) = \text{COL}(x + 41) \ldots \)

Use \( \text{COL}(x) = \text{COL}(x + 41) \) to finish the proof and find upper bound on \( W_3 \).
Think about this
Since $\text{COL}(x) = \text{COL}(x + 41) \ldots$

Use $\text{COL}(x) = \text{COL}(x + 41)$ to finish the proof and find upper bound on $W_3$. Think about this

$$\text{COL}(1) = \text{COL}(1+41) = \text{COL}(1+2\times41) = \cdots = \text{COL}(1+41\times41)$$
Since $\text{COL}(x) = \text{COL}(x + 41) \ldots$

Use $\text{COL}(x) = \text{COL}(x + 41)$ to finish the proof and find upper bound on $W_3$.
Think about this

$\text{COL}(1) = \text{COL}(1+41) = \text{COL}(1+2\times41) = \cdots = \text{COL}(1+41\times41)$

So 1 and $41^2$ are a square apart and the same color.
$W_3 \leq 1 + 41^2 = 1682$
Since \( \text{COL}(x) = \text{COL}(x + 41) \ldots \)

Use \( \text{COL}(x) = \text{COL}(x + 41) \) to finish the proof and find upper bound on \( W_3 \).
Think about this

\[
\text{COL}(1) = \text{COL}(1+41) = \text{COL}(1+2\times41) = \cdots = \text{COL}(1+41\times41)
\]

So 1 and \( 41^2 \) are a square apart and the same color.

\( W_3 \leq 1 + 41^2 = 1682 \)

Can we get better bound on \( W_3 \)?
Better Bound on $W_3$

Figure: If $x \geq 10$ then $\text{COL}(x) = \text{COL}(x + 7)$, so $W_3 \leq 59$
Reflection on $W_3$

1. Problem 5 (so hard) on UMCP HS Math Comp, 2006: Show that for all 3-colorings of $\{1, \ldots, 2006\}$ there exists 2 numbers that are a square apart that are the same color.

2. 240 took exam, 40 tried this problem, 10 got it right.


4. **Is there a HS-proof that $W_4$ exists?** Bill wanted to put this problem on the next HS exam to find out. He was (wisely) told **NO**.

5. The question still remains: Is there a HS proof that $W_4$ exists?
Reflection on $W_3$

1. Problem 5 (so hard) on UMCP HS Math Comp, 2006: Show that for all 3-colorings of \(\{1, \ldots, 2006\}\) there exists 2 numbers that are a square apart that are the same color.

2. 240 took exam, 40 tried this problem, 10 got it right.


4. **Is there a HS-proof that $W_4$ exists?** Bill wanted to put this problem on the next HS exam to find out. He was (wisely) told NO.

$W_4$ Exists: $\text{COL}(x) = \text{COL}(x + 290,085,290)$
Reflection on $W_4$

1. Zach’s proof shows $W_4 \leq 1 + 299,085,290^2$.
   PRO Proof is easy to verify
   CON Number is large, proof does not generalize to $W_5$.

2. The classical proof.
   PRO Gives bounds for $W_c$.
   CON Bounds are GINORMOUS, even for $W_2$.

3. A Computer Search showed that $W_4 = 58$.
   PRO Get exact value.
   CON not human-verifiable. Does not generalize to $W_5$.

Which do you prefer?
How Amazon Prices Books

April 19, 2020
Our Book on Amazon?

The book cost more used than new!
Our Book on Amazon?

The book cost more used than new!

Chapter 1 of

Problems with two Points:
More Explorations of Math and Computer Science

will be

The Mathematics of Book Pricing on Amazon