Ramsey’s Theorem
for the Infinite Complete Graph and
the Infinite Complete Hypergraph
Exposition by William Gasarch

1 Introduction

In this document we define notation for graphs and hypergraphs that we use
for the course and then look at Ramsey’s theorem and the Canonical Ramsey
theory on $\mathbb{N}$. Why start with $\mathbb{N}$? Because Joel Spencer said

Infinite Ramsey Theory is easier than Finite Ramsey Theory
because all of the messy constants go away.

2 Notation

Recall that a graph is a set of vertices and a set of edges which are unordered
pairs of vertices. Why pairs? We will generalize this by allowing edges to be
sets of size 1, 2 (the usual case), 3, general $a$ and not have any restriction on
size.

Notation 2.1

1. If $n \geq 1$ then $[n] = \{1, \ldots, n\}$.
2. If $a \in \mathbb{N}$ and $A$ is a set then $\binom{A}{a}$ is the set of all subsets of $A$ of size $a$.
   We commonly use $\binom{[n]}{a}$ and $\binom{\mathbb{N}}{a}$.

Def 2.2 Let $a \in \mathbb{N}$ (note that $a = 0$ is allowed). A $a$-hypergraph is a set of
vertices $V$ and a set of edges which is a subset of $\binom{V}{a}$.

Examples

1. A 0-hypergraph is just a set of vertices. This is just stupid but we’ll
   keep it around in case we need some edge case.

2. A 1-hypergraph is a set of vertices together with edges which are also
   vertices. So its just a set of vertices but some are also called edges.
3. A 2-hypergraph is the usual graphs you know and love.

4. A 3-hypergraph. Edges are sets of 3 vertices. \( V = \mathbb{N} \) and the edges are all \((a, b, c)\) such that \( a + b + c \equiv 0 \pmod{9} \). I could not have said \( a + 2b + 3c \equiv 0 \pmod{9} \) since then the order would matter. We are dealing with unordered hypergraphs. I could have said all \((a, b, c)\) with \( a < b < c \) such that \( a + 2b + 3c \equiv 0 \pmod{9} \).

5. Another example of a 3-hypergraph: let \( V \) be some set of points in the plane. Let the edges be all 3-sets of points that form non-degenerate triangles.

**Def 2.3** A hypergraph (notice the lack of a parameter) is a set of vertices \( V \) together with edges which are subsets of \( V \).

**Example**

1. \( V = \mathbb{N} \) and we take the set of all finite subsets of \( \mathbb{N} \) whose sum is \( \equiv 0 \pmod{9} \). Note that the empty set would be an edge.

2. \( V \) is a set of points in the plane. The edges are all of the lines in the plane.

3. Any \( a \)-hypergraph is also a hypergraph.

We are all familiar with the compete graph on \( \mathbb{N} \):

**Notation 2.4** \( K_\mathbb{N} \) is the graph \((V, E)\) where

\[
V = \mathbb{N} \\
E = \binom{\mathbb{N}}{2}
\]

Here is the complete \( a \)-hypergraph on \( \mathbb{N} \):

**Notation 2.5** \( K_{\mathbb{N}}^a \) is the hypergraph \((V, E)\) where

\[
V = \mathbb{N} \\
E = \binom{\mathbb{N}}{a}
\]

**Convention 2.6** In this course unless otherwise noted (1) a coloring of a graph is a coloring of the edges of the graph. and (2) a coloring of a hypergraph is a coloring of the edges of the hypergraph.
3 Ramsey Theory on the Complete 1-Hypergraph on \( \mathbb{N} \)

The following theorem is to obvious to prove but I want to state it:

**Theorem 3.1** For every 2-coloring of \( \mathbb{N} \) there is an infinite \( A \subseteq \mathbb{N} \) that is the same color.

Even though this is an easy theorem here are some questions:

1. Is there a finite version of this theorem?
2. If you are given a program that computes the coloring can you determine which color (or perhaps both) appears infinitely often?
3. What if you are given a simple computational device (e.g., a DFA with output). Then can you determine which color? What is the complexity of the problem?

What if I allow an infinite number of colors?

**Theorem 3.2** For every coloring of \( \mathbb{N} \) there is either (1) an infinite \( A \subseteq \mathbb{N} \) that is the same color, or (2) an infinite \( A \subseteq \mathbb{N} \) that all have different colors (called a rainbow set).

**Proof:** Let \( \text{COL} \) be a coloring of \( \mathbb{N} \). We define an infinite sequence of vertices,

\[
x_1, x_2, \ldots,
\]

and an infinite sequence of sets of vertices,

\[
V_0, V_1, V_2, \ldots,
\]

that are based on \( \text{COL} \).

Here is the intuition: Either \( \text{COL}(1) \) appears infinitely often (so we are done) or not. If not then we get rid of the finite number of vertices colored \( \text{COL}(1) \) except 1. We then do the same for \( \text{COL}(2) \). We will either find some color that appears infinitely often or create a sequence of all different colors.

We now describe it formally.
\(V_0 = \mathbb{N}\)
\(x_1 = 1\)

\(c_1 = \text{DONE if } |\{v \in V_0 \mid \text{COL}(v) = \text{COL}(x_1)\}| \text{ is infinite. And you are DONE! STOP}\)

\(= \text{COL}(x_1) \text{ otherwise}\)

\(V_1 = \{v \in V_0 \mid \text{COL}(v) \neq c_1\}\) (note that \(|V_1|\) is infinite)

Let \(i \geq 2\), and assume that \(V_{i-1}\) is defined. We define \(x_i\), \(c_i\), and \(V_i\):

\(x_i = \) the least number in \(V_{i-1}\)

\(c_i = \text{DONE if } |\{v \in V_{i-1} \mid \text{COL}(v) = \text{COL}(x_i)\}| \text{ is infinite. And you are DONE! STOP}\)

\(= \text{COL}(x_i) \text{ otherwise}\)

\(V_i = \{v \in V_{i-1} \mid \text{COL}(v) \neq c_i\}\) (note that \(|V_i|\) is infinite)

How long can this sequence go on for? If ever it stops then we are done as we have found a color appearing infinitely often. If not then the sequence

\[x_1, x_2, \ldots,\]

is infinite and each number in it is a different color, so we have found a rainbow set. \(\blacksquare\)

1. Is there a finite version of this theorem?

2. If you are given a program that computes the coloring can you determine which color (if any) appears infinitely often?

3. What if you are given a simple computational device (e.g., a DFA with output). Then can you determine which color? What is the complexity of the problem?

4 A Bit More Notation

For the case of the 1-hypergraph we didn’t need notions of complete graphs or homog sets, though that is what we were talking about. For \(a\)-hypergraphs we will.
**Def 4.1** Let $COL : \binom{\mathbb{N}}{2} \rightarrow [2]$. Let $V \subseteq \mathbb{N}$. The set $V$ is *homog* if there exists a color $c$ such that every elements of $\binom{V}{2}$ is colored $c$.

**Def 4.2** Let $COL : \binom{\mathbb{N}}{k} \rightarrow [c]$. Let $V \subseteq \mathbb{N}$. The set $V$ is *homog* if there exists a color $c$ such that every elements of $\binom{V}{k}$ is colored $c$.

## 5 Ramsey’s Theorem for the Infinite Complete Graphs

The following is Ramsey’s Theorem for $K_\mathbb{N}$.

**Theorem 5.1** *For every 2-coloring of the edges of $K_\mathbb{N}$ there is an infinite homog set.*

**Proof:** Let $COL$ be a 2-coloring of $K_\mathbb{N}$. We define an infinite sequence of vertices,

$$x_1, x_2, \ldots,$$

and an infinite sequence of sets of vertices,

$$V_0, V_1, V_2, \ldots,$$

that are based on $COL$.

Here is the intuition: Vertex $x_1 = 1$ has an infinite number of edges coming out of it. Some are RED, and some are BLUE. Hence there are an infinite number of RED edges coming out of $x_1$, or there are an infinite number of BLUE edges coming out of $x_1$ (or both). Let $c_1$ be a color such that $x_1$ has an infinite number of edges coming out of it that are colored $c_1$. Let $V_1$ be the set of vertices $v$ such that $COL(v, x_1) = c_1$. Then keep iterating this process.

We now describe it formally.

$$V_0 = \mathbb{N}$$

$$x_1 = 1$$

$$c_1 = \text{RED if } |\{v \in V_0 \mid COL(v, x_1) = \text{RED}\}| \text{ is infinite}$$

$$= \text{BLUE otherwise}$$

$$V_1 = \{v \in V_0 \mid COL(v, x_1) = c_1\} \text{ (note that } |V_1| \text{ is infinite})$$
Let $i \geq 2$, and assume that $V_{i-1}$ is defined. We define $x_i$, $c_i$, and $V_i$:

\[ x_i = \text{ the least number in } V_{i-1} \]

\[ c_i = \begin{cases} \text{RED} & |\{v \in V_{i-1} \mid \text{COL}(v, x_i) = \text{RED}\}| \text{ is infinite} \\ \text{BLUE} & \text{otherwise} \end{cases} \]

\[ V_i = \{v \in V_{i-1} \mid \text{COL}(v, x_i) = c_i\} \text{ (note that } |V_i| \text{ is infinite)} \]

(Note: look at the step where we define $c_i$. We are using the fact that if you 2-color $N$ you are guaranteed some color appears infinitely often; we are using the 1-hypergraph Ramsey Theorem. Later when we prove Ramsey on 3-hypergraphs we will use Ramsey on 2-hypergraphs.)

How long can this sequence go on for? Well, $x_i$ can be defined if $V_{i-1}$ is nonempty. We can show by induction that, for every $i$, $V_i$ is infinite. Hence the sequence

\[ x_1, x_2, \ldots \]

is infinite.

Consider the infinite sequence

\[ c_1, c_2, \ldots \]

Each of the colors in this sequence is either RED or BLUE. Hence there must be an infinite sequence $i_1, i_2, \ldots$ such that $i_1 < i_2 < \cdots$ and

\[ c_{i_1} = c_{i_2} = \cdots \]

Denote this color by $c$, and consider the vertices

\[ H = \{x_{i_1}, x_{i_2}, \cdots\} \]

We leave it to the reader to show that $H$ is homog.

**Exercise 1** Show that, for all $c \geq 3$, for every $c$-coloring of the edges of $K_N$, there is a an infinite homog set.

Questions to ponder:

1. Is there a finite version?
2. What if we allow an infinite number of colors?
6 First “Application”

We will prove a theorem that is well known; however, this proof is by Gasarch from January 2017.

**Theorem 6.1** Let $d \in \mathbb{N}$, $d \geq 1$. If $p_1, p_2, \ldots$ is an infinite set of points in $\mathbb{R}^d$. There exists a subsequence $q_1, q_2, \ldots$ such that, restricted to any coordinate, the sequence will be either strictly increasing, strictly decreasing, or constant.

**Proof:** We do the proof in $\mathbb{R}^2$ but all of the ideas are the same for $\mathbb{R}^d$.

We define the following 9-coloring of pairs of points: Let $p_i = (x_i, y_i)$. We assume $i < j$. Then

$$
COL(p_i, p_j) = \begin{cases} 
(DEC, DEC) & \text{if } x_i > x_j \text{ and } y_i > y_j \\
(DEC, CON) & \text{if } x_i > x_j \text{ and } y_i = y_j \\
(DEC, INC) & \text{if } x_i > x_j \text{ and } y_i < y_j \\
(CON, DEC) & \text{if } x_i = x_j \text{ and } y_i > y_j \\
(CON, CON) & \text{if } x_i = x_j \text{ and } y_i = y_j \\
(CON, INC) & \text{if } x_i = x_j \text{ and } y_i < y_j \\
(INC, DEC) & \text{if } x_i < x_j \text{ and } y_i > y_j \\
(INC, CON) & \text{if } x_i < x_j \text{ and } y_i = y_j \\
(INC, INC) & \text{if } x_i < x_j \text{ and } y_i < y_j 
\end{cases}
$$

(1)

Take the homog set. Clearly it will be, in each coordinate, decreasing, constant, or increasing.

For $\mathbb{R}^d$ you would use the $3^d$-coloring. \[\square\]

Here is what is probably the classical proof (though I never saw the theorem until I proved it myself).

First prove the theorem for $d = 1$ then do an induction on $d$. The induction step is easy; however how do do the $d = 1$ case? Let $p_1, p_2, \ldots$ be a sequence of reals.

1) First Alternative Proof: Use Ramsey’s theorem on pairs of numbers, the proof above but for $d = 1$. The good news- we only need to use Ramsey for 3-colors. The bad news- we are looking for non-ramsey proofs.

2) Second Alternative Proof:
There are some very easy cases whose proofs we omit and then one hard case:

1. The function \( f(i) = \max\{p_1, \ldots, p_i\} \) goes to infinity. This case is easy and we leave it to you.

2. The function \( f(i) = \max\{p_1, \ldots, p_i\} \) goes to negative infinity. This case is easy and we leave it to you.

3. Neither (1) nor (2) happens. Hence there exists reals \( a < b \) such that \( p_1, p_2, \ldots \in [a, b] \). This case we do below.

By the Bolzano-Weierstrass theorem every sequence of reals in a closed interval has a limit point (there may many limit points but we just need 1). Let \( p \) be a limit point of \( p_1, p_2, \ldots \). There are three cases:

1. \((\forall n)(\exists i \geq n)[p_i = p]\). The sequence has a constant subsequence. Hence there is a constant subsequence and you are done.

2. \((\forall n)(\exists i \geq n)[0 < p - p_i < \frac{1}{n}]\). Hence there is an increasing subsequence and you are done.

3. \((\forall n)(\exists i \geq n)[0 < p_i - p < \frac{1}{n}]\). Hence there is a decreasing increasing subsequence and you are done.

The proof above uses Ramsey’s theorem a little bit and perhaps a lot. The splitting into three cases can be regraded as using Ramsey on 1-hypergraphs. This is minor- its just the Pigeon hole principle really, and nobody in math every says Hey! I’m using Ramsey Theory! if they are just using that principle. More seriously- look at the proof of the BW theorem – some say it is Ramsey-like.

Also, see www.youtube.com/watch?v=dfO18klwKHget for a rap song about the BW theorem. Really!

7 Ramsey’s Theorem for 3-Hypergraphs: First Proof

**Theorem 7.1** For all \( \text{COL} : (\mathbb{N}^3 \to 2 \) there exists an infinite 3-homog set.
Proof:

CONSTRUCTION

PART ONE

\[ V_0 = \mathbb{N} \]
\[ x_0 = 1. \]
Assume \( x_1, \ldots, x_{i-1} \) defined, \( V_{i-1} \) defined and infinite.

\[ x_i = \text{the least number in } V_{i-1} \]
\[ V_i = V_{i-1} - \{x_i\} \text{ (Will change later without changing name.)} \]
\[ COL^*(x, y) = COL(x_i, x, y) \text{ for all } \{x, y\} \in \binom{V_i}{2} \]
\[ V_i = \text{an infinite 2-homogeneous set for } COL^* \]
\[ c_i = \text{the color of } V_i \]

KEY: for all \( y, z \in V_i \), \( COL(x_i, y, z) = c_i \).

END OF PART ONE

PART TWO

We have vertices

\[ x_1, x_2, \ldots, \]

and associated colors

\[ c_1, c_2, \ldots. \]

There are only two colors, hence, by the 1-homog Ramsey Theorem there exists \( i_1, i_2, \ldots \), such that \( i_1 < i_2 < \ldots \) and

\[ c_{i_1} = c_{i_2} = \cdots \]

We take this color to be RED. Let

\[ H = \{x_{i_1}, x_{i_2}, \ldots\}. \]

We leave it to the reader to show that \( H \) is homog.

END OF PART TWO

END OF CONSTRUCTION

Exercise 2
1. Show that, for all $c$, for all $c$-coloring of $K_N^3$ there exists an infinite 3-homog set.

2. State and prove a theorem about $c$-coloring $\binom{N}{3}$.

3. What if we allow an infinite number of colors?

8 Ramsey’s Theorem for 3-Hypergraphs: Second Proof

In the last section the proof went as follows:

- Color a node by using 2-hypergraph Ramsey. This is done $\omega$ times.
- After the nodes are colored we use 1-hypergraph. This is done once.

We given an alternative proof where:

- Color an edge by using 1-hypergraph Ramsey. This is done $\omega$ times.
- After all the edges of an infinite complete graph are colored we use 2-hypergraph Ramsey. This is done once.

We now proceed formally.

**Theorem 8.1** For all $COL : \binom{N}{3} \rightarrow [2]$ there exists an infinite 3-homog set.

**Proof:**

Let $COL$ be a 2-coloring of $\binom{N}{3}$. We define a sequence of vertices,

$$x_1, x_2, \ldots,$$

Here is the intuition: Let $x_1 = 1$ and $x_2 = 2$. The vertices $x_1, x_2$ induces the following coloring of $\{3, 4, \ldots\}$.

$$COL^*(y) = COL(x_1, x_2, y).$$

Let $V_1$ be an infinite 1-homogeneous for $COL^*$. Let $COL^{**}(x_1, x_2)$ be the color of $V_1$. Let $x_3$ be the least vertex left (bigger than $x_2$).
The number $x_3$ induces two colorings of $V_1 - \{x_3\}$:

$$(\forall y \in V_1 - \{x_3\})[COL_1(y) = COL(x_1, x_3, y)]$$

$$(\forall y \in V_1 - \{x_3\})[COL_2(y) = COL(x_2, x_3, y)]$$

Let $V_2$ be an infinite 1-homogeneous for $COL_1^*$. Let $COL_2^*(x_1, x_3)$ be the color of $V_2$. Restrict $COL_2^*$ to elements of $V_2$, though still call it $COL_2^*$. We reuse the variable name $V_2$ to be an infinite 1-homogeneous for $COL_2^*$. Let $COL_2^*(x_1, x_3)$ be the color of $V_2$. Let $x_4$ be the least element of $V_2$. Repeat the process.

We describe the construction formally.

**CONSTRUCTION**

**PART ONE:**

Let $i \geq 2$. Assume that $x_1, \ldots, x_{i-1}, V_{i-1}$, and $COL^*(\{x_1, \ldots, x_{i-1}\}) \to \{\text{RED}, \text{BLUE}\}$ are defined.

1. $x_i = \text{the least element of } V_{i-1}$
2. $V_i = V_{i-1} - \{x_i\}$ (We will change this set without changing its name).

We define $COL_2^*(x_1, x_i), COL_2^*(x_2, x_i), \ldots, COL_2^*(x_{i-1}, x_i)$. We will also define smaller and smaller sets $V_i$ (not smaller by size – they are all infinite – but smaller by being subsets). We will keep the variable name $V_i$ throughout.

For $j = 1$ to $i - 1$

1. $COL_j^*: V_i \to \{\text{RED}, \text{BLUE}\}$ is defined by $COL_j^*(y) = COL(x_j, x_i, y)$.
2. Let $V_i$ be redefined as an infinite 1-homogeneous set for $COL^*$. Note that $V_i$ is still infinite.
3. $COL_2^*(x_j, x_i)$ is the color of $V_i$. 

PART TWO:

From PART ONE we have a set of vertices $X$

$$X = \{x_1, x_2, \ldots \}$$

and a 2-coloring $COL^{**}$ of $\binom{X}{2}$. By the 2-hypergraph Ramsey Theorem there exists an infinite homog (with respect to $COL^{**}$) set

$$H = \{y_1, y_2, \ldots \}$$

Assume that the homog color is $R$. Then for $i < j < k$

$$COL(y_i, y_j, y_k) = COL^{**}(y_i, y_j) = R$$

So $H$ is homog for $COL$. END OF PART TWO

END OF CONSTRUCTION