Proof of Infinite Canonical Ramsey's Theorem
That Does Not Use Hypergraph Ramsey
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1 Intuition

We will proof the Infinite Canonical Ramsey theory (for graphs) but not use any hypergraph Ramsey Theorem.

It will be close in spirit to the proof of the infinite Ramsey Theorem.

We first restate how we used the infinite 1-hypergraph Ramsey Theorem to prove the 2-hypergraph Ramsey Theorem:

If $\binom{\mathsf{N}}{2}$ is 2-colored and there is an infinite sequence of vertices:

$$X = \{x_1, x_2, x_3, \ldots\}$$

Then either

- There exists infinite $Y_R \subseteq X$ such that

$$(\forall x \in Y_R)[COL(x, y) = R].$$

- There exists infinite $Y_B \subseteq X$ such that

$$(\forall x \in Y_B)[COL(x, y) = B].$$

We then replace X with Y_R or Y_B.

We now describe the analog of that process which we will be using to prove 2-hypergraph Can Ramsey from 1-hypergraph Can Ramsey.

If $\binom{\mathsf{N}}{2}$ is colored (note no bound on the number of colors) and there is an infinite sequence of vertices:

$$x_1, x_2, x_3, \ldots$$

Then either

- There exists color c and infinite $Y_c \subseteq X$ such that

$$(\forall x \in Y)[COL(x, y) = c].$$

- There exists infinite $Y_\omega \subseteq X$ such that,

$$(\forall c)(\exists! y \in Y)[COL(x,y) = c].$$

  (Notation -($\exists! y$) means there is ONE y.)

We then replace X with Y_1 or Y_2 or $\cdots$ or Y_ω.

2 Ramsey Theory on the Complete 1-Hypergraph on N

The following theorem is to obvious to prove but I want to state it:

Theorem 2.1 *For every 2-coloring of N there is an infinite $A \subseteq \mathsf{N}$ that is the same color.*

Even though this is an easy theorem here are some questions:

1. Is there a finite version of this theorem?
2. If you are given a program that computes the coloring can you determine which color (or perhaps both) appears infinitely often?
3. What if you are given a simple computational device (e.g., a DFA with output). Then can you determine which color? What is the complexity of the problem?

What if I allow an infinite number of colors?

Theorem 2.2 *For every coloring of N there is either (1) an infinite $A \subseteq \mathsf{N}$ that is the same color, or (2) an infinite $A \subseteq \mathsf{N}$ that all have different colors (called a rainbow set).*

Proof: Let COL be a coloring of N. We define an infinite sequence of vertices,

$$x_1, x_2, \ldots,$$

and an infinite sequence of sets of vertices,

$$V_0, V_1, V_2, \ldots,$$

that are based on COL.

Here is the intuition: Either

- There is some color c such that $COL(1,x)$ is c infnitely often. Then restrict to that set and color 1 with (H,c).
- For every color c the set of x with $COL(1,x)=c$ is finite. Then thin out the set so that $COL(1,x_2)$, $COL(1,x_3)$, etc are all different. (When dealing with x_2 or x_3 later instead of x_1 this will get more complicated.)

We now describe it formally.

$$\begin{aligned} V_0 = & \quad \mathsf{N} \\ x_1 = & \quad 1 \end{aligned}$$

If $(\exists c)[|\{v \in V_0 \mid COL(x_1,v)=c\}|=\omega$ then:

- $c_1 = (H,c)$
- $V_1 = \{v \in V_0 \mid COL(x_1,v)=c\}$. (Note that V_1 is infinite)

If $(\forall c)|\{v \in V_0 \mid COL(x_1,v)=c\}| < \omega$ then:

- $V_1 = \{v \in V_0 \mid (\exists c)[COL(x_1,v)=c \wedge (\forall x_1 < u < v)[COL(x_1,u) \neq c]\}$ (so v is the first first with $COL(x_1,v)=c$. Hence there will only be ONE v with $COL(x_1,v)=c$.) (Note that V_1 is infinite)
- $c_1 = (RB,1)$. (The 1 only marks that this is the first rainbow-color assigned.)

Let $i \geq 2$, and assume that V_{i-1} is defined. We define x_i, c_i, and V_i:
x_i is the least element of V_{i-1} bigger than x_{i-1}.
If $(\exists c)[|\{v \in V_{i-1} \mid COL(x_i,v)=c\}|=\omega$ then:

- $c_i = (H,c)$
- $V_i = \{v \in V_0 \mid COL(x_i,v)=c\}$. (Note that V_i is infinite)

If $(\forall c)|\{v \in V_{i-1} \mid COL(x_i,v)=c\}| < \omega$ then:

- $V_i = \{v \in V_{i-1} \mid (\exists c)[COL(x_i,v)=c \wedge (\forall x_i < u < v)[COL(x_i,u) \neq c]\}$ (so v is the first first with $COL(x_i,v)=c$. Hence there will only be ONE v with $COL(x_i,v)=c$.) (Note that V_i is infinite)
- For $1 \leq j \leq i-1$ we need to see how V_i interacts with V_j. Hence we do the following.

– If

Let $i \geq 2$, and assume that V_{i-1} is defined. We define x_i, c_i, and V_i:

How long can this sequence go on for? If ever it stops then we are done as we have found a color appearing infinitely often. If not then the sequence

$$x_1, x_2, \ldots,$$

is infinite and each number in it is a different color, so we have found a rainbow set. ∎

1. Is there a finite version of this theorem?
2. If you are given a program that computes the coloring can you determine which color (if any) appears infinitely often?
3. What if you are given a simple computational device (e.g., a DFA with output). Then can you determine which color? What is the complexity of the problem?

3 Can Ramsey's Theorem for the Infinite Complete Graphs

Theorem 3.1 *For every coloring of the edges of K_{N} there is either an an infinite homog set, an infinite min-homog set, an infinite max-homog set, or an infinite rainbow set.*

Proof: Let COL be a coloring of K_{N}. We define an infinite sequence of vertices,

$$x_1, x_2, \ldots,$$

and an infinite sequence of sets of vertices,

$$V_0, V_1, V_2, \ldots,$$

that are based on COL. We will also color the vertices as we process them. The colors will be

(H, c) to indicate that we used Y_c above.

(RB, i) to indicate we did Y_ω. If two vertices have the same (RB, i) it means that they agree on all vertices past the largest of the two.

CONSTRUCTION

PART ONE

$$V_0 = \mathsf{N}$$

Let $i \geq 1$, and assume that V_{i-1} is defined. We define x_i, c_i, and V_i: We use the Y_c, Y_ω notation above.

x_i gets the least element of V_{i-1}.

If there exists c such that Y_c is infinite then

$$\begin{aligned} c_i &= (H, c) \\ V_i &= Y_c \end{aligned}$$

If no such c exist then there exists Y_ω. We initially take

$V_i = Y_\omega$

But we may thin it out. And we haven't colored x_i yet.

Do the following:

For all $1 \leq j \leq i-1$ such that $COL(x_j) = (RB, k)$ for some k then:

1. If $|\{y \in Y_\omega : COL(x_j, y) = COL(x_i, y)\}| = \omega$ then let V_i be this set and let $c_i = c_j$. (So $COL(x_i)$ will be of the form (RB, k) for some k). You are done and do not go to the next j.

2. If $|\{y \in Y_\omega : COL(x_j, y) = COL(x_i, y)\}| < \omega$ then let V_i be the Y_ω minus those vertices.

If Case 1 ever happens then we are done. If Case 2 always happens then note that x_i disagrees with every x_j on every element $> x_i$. We c_i with (RB, k) where k is the least number not used for a rainbow color yet.

END OF PART ONE

PART TWO

Consider the infinite sequence

$$c_1, c_2, \ldots$$

There are several cases:

- There is a c such that (H, c) appears infinitely often. Let

  $$H = \{x_i : c_i = (H, c)\}.$$

  This set is infinite homog.

- There is an infinite number of vertices colored H. Let

$$H' = \{x_i : (\exists c)[c_i = (H, c)\}$$

By the 1-hypergraph Can Ramsey applied to the coloring $COL(x_i) = c$, and the premise, we get a set H which we renumber so that

$$H = \{y_1 < y_2 < y_3 < \cdots\}$$

and $COL(y_i) = (H, i)$. H is infinite min-homog.

- There is an k such that (RB, k) appears infinitely often. Let

$$H = \{x_i : c_i = (RB, k)\}.$$

This set is infinite max-homog.

- There is an infinite number of vertices colored RB. Let

$$H' = \{x_i : (\exists k)[c_i = (RB, k)\}$$

By the 1-hypergraph Can Ramsey applied to the coloring $COL(x_i) = k$, and the premise, we get a set H which we renumber so that

$$H = \{y_1 < y_2 < y_3 < \cdots\}$$

and $COL(y_i) = (RB, i)$. H is infinite rainbow.

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